in time; that is, the ratio

$$\frac{\exp\,[G(x^0,r_1)]}{\exp\,[G(x^0,r_2)]} \tag{12.26}$$

must be independent of x^0. Therefore one must have

$$G(x^0,r_1) = G(x^0,r_2) + F(r_1,r_2) \tag{12.27}$$

If we now choose a fixed value of r_2, we can write

$$G(x^0,r_1) = g(x^0) + f(r_1) \tag{12.28}$$

We can therefore obtain the following form of the line element (12.25):

$$ds^2 = (dx^0)^2 - e^{g(x^0)+f(r)}\,d\sigma^2 \tag{12.29}$$

which is based entirely on symmetry arguments. The functions f and g which appear in the exponent are arbitrary (g is not to be confused with the determinant of $g_{\mu\nu}$).

To investigate the form of ds^2 still further, we consider now the Einstein equations (12.4). In order to calculate the Ricci tensor which enters into (12.4), we have to determine the Christoffel symbols of the metric (12.29). This is a straightforward calculation, but for the convenience of the reader, we shall indicate briefly the main steps in this computation. The geodesics of the metric (12.29) satisfy the variational condition

$$\delta\int[(\dot{x}^0)^2 - e^G(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\varphi}^2)]\,ds = 0 \tag{12.30}$$

if we use the coordinates x^0, r, θ, and φ. We obtain, therefore, the following Euler-Lagrange equations:

$$\begin{aligned}
&\ddot{x}^0 + \tfrac{1}{2}g'e^G(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\varphi}^2) = 0\\
&\ddot{r} + \tfrac{1}{2}f'\dot{r}^2 + g'\dot{x}^0\dot{r} - \left(\tfrac{1}{2}f' + \frac{1}{r}\right)(r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\varphi}^2) = 0\\
&\ddot{\theta} + 2\left(\tfrac{1}{2}f' + \frac{1}{r}\right)\dot{r}\dot{\theta} + g'\dot{x}^0\dot{\theta} - \sin\theta\cos\theta\,\dot{\varphi}^2 = 0\\
&\ddot{\varphi} + 2\left(\tfrac{1}{2}f' + \frac{1}{r}\right)\dot{r}\dot{\varphi} + g'\dot{x}^0\dot{\varphi} + 2\cot\theta\,\dot{\theta}\dot{\varphi} = 0
\end{aligned} \tag{12.31}$$

Here the dot denotes differentiation with respect to the parameter s, while $g'(x^0)$ and $f'(r)$ denote the derivatives of $g(x^0)$ and $f(r)$ with respect to their single arguments.

From (12.31) we can read off the list of nonvanishing Christoffel symbols in the metric (12.29). We find that

$$\begin{aligned}
&\left\{{0 \atop 1\;1}\right\} = \tfrac{1}{2}g'e^G \qquad \left\{{0 \atop 2\;2}\right\} = \tfrac{1}{2}g'e^Gr^2 \qquad \left\{{0 \atop 3\;3}\right\} = \tfrac{1}{2}g'e^Gr^2\sin^2\theta\\
&\left\{{1 \atop 0\;1}\right\} = \tfrac{1}{2}g' \qquad \left\{{1 \atop 1\;1}\right\} = \tfrac{1}{2}f' \qquad \left\{{1 \atop 2\;2}\right\} = -r^2\left(\tfrac{1}{2}f' + \frac{1}{r}\right)\\
&\left\{{1 \atop 3\;3}\right\} = -r^2\left(\tfrac{1}{2}f' + \frac{1}{r}\right)\sin^2\theta\\
&\left\{{2 \atop 0\;2}\right\} = \tfrac{1}{2}g' \qquad \left\{{2 \atop 1\;2}\right\} = \left(\tfrac{1}{2}f' + \frac{1}{r}\right) \qquad \left\{{2 \atop 3\;3}\right\} = -\sin\theta\cos\theta\\
&\left\{{3 \atop 0\;3}\right\} = \tfrac{1}{2}g' \qquad \left\{{3 \atop 1\;3}\right\} = \left(\tfrac{1}{2}f' + \frac{1}{r}\right) \qquad \left\{{3 \atop 2\;3}\right\} = \cot\theta
\end{aligned} \tag{12.32}$$

From (12.29) we find for the determinant g of the metric tensor

$$\log\sqrt{-g} = \tfrac{3}{2}g(x^0) + \tfrac{3}{2}f(r) + 2\log r + \log|\sin\theta| \tag{12.33}$$

From the above it is then easy to see that

$$\begin{aligned}
&\left\{{\rho \atop 0\;0}\right\}_{|\rho} = 0 \qquad \left\{{\rho \atop 1\;1}\right\}_{|\rho} = \tfrac{1}{2}e^G(g'' + g'^2) + \tfrac{1}{2}f''\\
&\left\{{\rho \atop 2\;2}\right\}_{|\rho} = \left[\tfrac{1}{2}e^G(g'' + g'^2) - \left(\tfrac{1}{2}f'' + \frac{1}{r}f' + \frac{1}{r^2}\right)\right]r^2\\
&\left\{{\rho \atop 3\;3}\right\}_{|\rho} = \left[\tfrac{1}{2}e^G(g'' + g'^2) - \left(\tfrac{1}{2}f'' + \frac{1}{r}f'\right)\right]r^2\sin^2\theta - \cos^2\theta
\end{aligned} \tag{12.34}$$

which will be very useful shortly. We also need the following relations, which can be obtained from (12.32):

$$\begin{aligned}
&\left\{{\alpha \atop 0\;\rho}\right\}\left\{{\rho \atop 0\;\alpha}\right\} = \tfrac{3}{4}g'^2\\
&\left\{{\alpha \atop 1\;\rho}\right\}\left\{{\rho \atop 1\;\alpha}\right\} = \tfrac{1}{2}e^Gg'^2 + \tfrac{3}{4}f'^2 + \frac{2}{r}f' + \frac{2}{r^2}\\
&\left\{{\alpha \atop 2\;\rho}\right\}\left\{{\rho \atop 2\;\alpha}\right\} = \left[\tfrac{1}{2}e^Gg'^2 - \tfrac{1}{2}f'^2 - \frac{2}{r}f' - \frac{2}{r^2} + \frac{1}{r^2}\cot^2\theta\right]r^2\\
&\left\{{\alpha \atop 3\;\rho}\right\}\left\{{\rho \atop 3\;\alpha}\right\} = \left[\tfrac{1}{2}e^Gg'^2 - \tfrac{1}{2}f'^2 - \frac{2}{r}f' - \frac{2}{r^2} - \frac{2}{r^2}\cot^2\theta\right]r^2\sin^2\theta
\end{aligned} \tag{12.35}$$

This completes the chore of obtaining the various terms we shall need in expressing the field equations in terms of the functions f and g.

The most convenient way now to write the contracted Riemann tensor is to use the definition of $R_{\mu\nu}$ in (5.119) and the expression (3.11) for the contracted Christoffel symbol, $\left\{ {\alpha \atop \beta\ \alpha} \right\} = (\log \sqrt{-g})_{|\beta}$, to obtain

$$(12.36)\quad R_{\mu\nu} = \left\{ {\alpha \atop \mu\ \alpha} \right\}_{|\nu} - \left\{ {\rho \atop \mu\ \nu} \right\}_{|\rho} + \left\{ {\alpha \atop \rho\ \nu} \right\} \left\{ {\rho \atop \mu\ \alpha} \right\} - \left\{ {\alpha \atop \rho\ \alpha} \right\} \left\{ {\rho \atop \mu\ \nu} \right\}$$
$$= (\log \sqrt{-g})_{|\mu|\nu} - \left\{ {\rho \atop \mu\ \nu} \right\}_{|\rho} + \left\{ {\alpha \atop \rho\ \nu} \right\} \left\{ {\rho \atop \mu\ \alpha} \right\} - \left\{ {\rho \atop \mu\ \nu} \right\} (\log \sqrt{-g})_{|\rho}$$

Substitution of the various terms found in (12.32) to (12.35) into the expression (12.36) for $R_{\mu\nu}$ then yields

$$(12.37)\quad \begin{aligned} R_{00} &= \tfrac{3}{2}g'' + \tfrac{3}{4}g'^2 \\ R_{11} &= f'' + \frac{1}{r}f' - e^G(\tfrac{1}{2}g'' + \tfrac{3}{4}g'^2) \\ R_{22} &= r^2\left[\tfrac{1}{2}f'' + \tfrac{1}{4}f'^2 + \frac{3}{2}\frac{1}{r}f' - e^G(\tfrac{1}{2}g'' + \tfrac{3}{4}g'^2)\right] \\ R_{33} &= r^2\sin^2\theta\left[\tfrac{1}{2}f'' + \tfrac{1}{4}f'^2 + \frac{3}{2}\frac{1}{r}f' - e^G(\tfrac{1}{2}g'' + \tfrac{3}{4}g'^2)\right] \end{aligned}$$

By analogous calculations it is easily seen that $R_{\mu\nu} = 0$ if $\mu \neq \nu$. This was, of course, to be expected in view of the symmetries of the line element (12.29).

To obtain the mixed tensor $R^{\mu}{}_{\nu}$ we need only raise one index with $g^{\mu\nu}$, the inverse tensor to $g_{\mu\nu}$. From (12.29) these tensors are easily seen to be

$$(12.38)\quad g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -e^G & 0 & 0 \\ 0 & 0 & -e^G r^2 & 0 \\ 0 & 0 & 0 & -e^G r^2 \sin^2\theta \end{pmatrix}$$

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -e^{-G} & 0 & 0 \\ 0 & 0 & -\dfrac{e^{-G}}{r^2} & 0 \\ 0 & 0 & 0 & -\dfrac{e^{-G}}{r^2\sin^2\theta} \end{pmatrix}$$

Thus from (12.37) we obtain the diagonal mixed tensor $R^{\mu}{}_{\nu}$:

$$(12.39)\quad \begin{aligned} R^0{}_0 &= \tfrac{3}{2}g'' + \tfrac{3}{4}g'^2 \\ R^1{}_1 &= (\tfrac{1}{2}g'' + \tfrac{3}{4}g'^2) - e^{-G}\left(f'' + \frac{1}{r}f'\right) \\ R^2{}_2 = R^3{}_3 &= (\tfrac{1}{2}g'' + \tfrac{3}{4}g'^2) - e^{-G}\left(\tfrac{1}{2}f'' + \tfrac{1}{4}f'^2 + \frac{3}{2}\frac{1}{r}f'\right) \end{aligned}$$

Contraction of $R^{\mu}{}_{\nu}$ then leads to the scalar R:

$$(12.40)\quad R = 3(g'' + g'^2) - 2e^{-G}\left(f'' + \tfrac{1}{4}f'^2 + \frac{2}{r}f'\right)$$

In (12.39) and (12.40) we have all the quantities which enter the field equations. Thus we finally arrive at the explicit form of the Einstein field equations (12.4) in terms of f and g:

$$(12.41)\quad \begin{aligned} -\frac{8\pi\kappa}{c^2}T^0{}_0 &= \left[e^{-G}\left(f'' + \frac{f'^2}{4} + \frac{2f'}{r}\right) - \tfrac{3}{4}g'^2\right] + \Lambda \\ -\frac{8\pi\kappa}{c^2}T^1{}_1 &= \left[e^{-G}\left(\frac{f'^2}{4} + \frac{f'}{r}\right) - g'' - \tfrac{3}{4}g'^2\right] + \Lambda \\ -\frac{8\pi\kappa}{c^2}T^2{}_2 = -\frac{8\pi\kappa}{c^2}T^3{}_3 &= \left[e^{-G}\left(\frac{f''}{2} + \frac{f'}{2r}\right) - g'' - \tfrac{3}{4}g'^2\right] + \Lambda \\ -\frac{8\pi\kappa}{c^2}T^{\mu}{}_{\nu} &= 0 \qquad \text{for } \mu \neq \nu \end{aligned}$$

These will be useful in further simplifying the line element and also in explicitly solving the field equations, as we shall show in Chap. 13.

Let us discuss next the consequences of the condition of local isotropy on the form of the energy-momentum tensor $T^{\mu}{}_{\nu}$. We shall use a coordinate system in which the spatial line element is proportional to

$$d\sigma^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2$$

If we subject the neighborhood of the point considered to an orthogonal transformation of the space coordinates, the individual space components $T^m{}_n$ of the energy-momentum tensor should not change because of the assumed invariance of the energy-momentum distribution under such rotation. Since the matrix $T^m{}_n$ is invariant under orthogonal transformation, all its eigenvalues must be equal and it must be a multiple of the unit matrix; that is,

$$(12.42)\quad T^m{}_n = A\,\delta^m{}_n$$

where A is a scalar which may depend on x^0 and r. But since $\delta^m{}_n = g^m{}_n$, we have in (12.42) a tensor equation which holds in every coordinate system in three-dimensional space. In particular, if we use the polar coordinates r, θ, and φ, we also have

$$T^i{}_k = A g^i{}_k = A\,\delta^i{}_k \tag{12.43}$$

Thus, for every coordinate system,

$$T^1{}_1 = T^2{}_2 = T^3{}_3 \tag{12.44}$$

Inserting this into (12.41), we obtain the following relation on f:

$$f'' - \tfrac{1}{2}(f')^2 - \frac{1}{r}f' = 0 \tag{12.45}$$

This equation admits the first integral

$$f' = are^{f/2} \tag{12.46}$$

which leads to the general solution

$$e^f = \frac{b^2}{[1 - (ab/4)r^2]^2} \tag{12.47}$$

where a and b are arbitrary constants. When writing out the metric form, we can absorb the constant b^2 into the function $e^{g(x^0)}$ and define a new arbitrary constant by

$$|ab| = \frac{1}{r_0^2} \tag{12.48}$$

Then we obtain

$$ds^2 = (dx^0)^2 - e^{g(x^0)}\frac{1}{\left(1 + \dfrac{k}{4}\dfrac{r^2}{r_0^2}\right)^2}\,d\sigma^2 \tag{12.49}$$

where $k = 0, +1, -1$, corresponding to $ab = 0$, ab negative, and ab positive. These values describe a Euclidean space, a spherical space, and a pseudo-spherical space, as we shall presently discuss.

We can gain somewhat more insight into the nature of the Riemann space defined by the metric (12.49). From (12.41) and (12.44) it is evident that

$$G^1{}_1 = G^2{}_2 = G^3{}_3 = e^{-f}\left(\tfrac{1}{4}f'^2 + \frac{1}{r}f'\right)e^{-g} - g'' - \tfrac{3}{4}g'^2 \tag{12.50}$$

where we remind the reader that $g = g(x^0)$. On the other hand, a simple calculation with the explicit form (12.47) for e^f yields

$$e^{-f}\left(\tfrac{1}{4}f'^2 + \frac{1}{r}f'\right) = \frac{a}{b} = \text{const} \tag{12.51}$$

We find, therefore, not only that the space part of $G^\mu{}_\nu$ is isotropic,

$$G^1{}_1 = G^2{}_2 = G^3{}_3$$

but that it is constant throughout three-space at any given moment x^0. Indeed, from (12.50) and (12.51), we can say explicitly that

$$G^1{}_1 = G^2{}_2 = G^3{}_3 = e^{-g}\left(\frac{a}{b}\right) - g'' - \tfrac{3}{4}g'^2 \qquad g = g(x^0) \tag{12.52}$$

This demonstrates the complete homogeneity of three-space despite the apparent distinction of the center of the spherical coordinate system.

The form of line element (12.49) defines the so-called Robertson-Walker metric (Robertson, 1935). As we have shown above, it is characterized by the purely geometric property $G^1{}_1 = G^2{}_2 = G^3{}_3$. Thus, although we were guided by reasoning on the tensor $T^\mu{}_\nu$, we have arrived at a line element with a characteristic geometric structure which is interesting in its own right. It is therefore not surprising that the Robertson-Walker line element has been studied extensively in pure differential geometry theory. In the non-Euclidean geometries of the nineteenth century, the space elements of the Robertson-Walker metric played an important role. The case $k = -1$ occurred in the non-Euclidean geometry of Bolyai and Lobachevski, and the case $k = +1$ was first extensively discussed by Riemann. While the main interest of these mathematicians at that time centered around the problem of parallels in geometry, their basic geometric axioms led them to such high symmetry assumptions on the space considered that they were forced into the line element of (12.49). While the assumption on geometric isotropy of space is based on the principle of sufficient reason, that is, the argument that we do not know any reason why certain space directions should be distinguished from others, the isotropy of the distribution of matter is an observational fact. Thus, from the methodological point of view, Mach's principle plays a decisive role in establishing the Robertson-Walker line element in cosmology. The ultimate determining factor for geometry is the empirical law of matter distribution in the universe, as we have tried to show above.

12.4 Further Properties of the Robertson-Walker Metric

In order to study the dynamics of material particles in the Robertson-Walker metric, we need to determine their possible trajectories, i.e., the geodesics corresponding to (12.49). These are defined by

$$\frac{d^2x^\alpha}{ds^2} + \begin{Bmatrix} \alpha \\ \beta\ \gamma \end{Bmatrix} \frac{dx^\beta}{ds}\frac{dx^\gamma}{ds} = 0 \tag{12.53}$$

The most natural and important of these trajectories in the present case are those which describe a material particle with fixed space coordinates. As we stated in the preceding section, our distinguished three-space is characterized by the fact that in it matter is on the average at rest; thus the name co-moving coordinate system. It is therefore important that we verify that all curves $x^1 = \text{const}$, $x^2 = \text{const}$, $x^3 = \text{const}$ are indeed solutions of (12.52). It is evident that we need only show that the Christoffel symbol

$$\begin{Bmatrix} i \\ 0\ 0 \end{Bmatrix} = \frac{g^{i\rho}}{2}(g_{0\rho|0} + g_{\rho 0|0} - g_{00|\rho}) \tag{12.54}$$

is zero; this, however, is immediately obvious from the form of the line element (12.49).

We have therefore verified that the Robertson-Walker metric represents a world in which a homogeneous distribution of matter is anchored to a co-moving coordinate system. On the other hand, a small test particle can certainly move along other geodesics, i.e., those for which ds need not be dx^0. Indeed, the theory of the initial-value problems for ordinary differential equations shows clearly that, if (12.52) has a solution such that all $\dot{x}^i$ vanish for a given moment x^0, we shall have $\dot{x}^i \equiv 0$ for all time. Conversely, if a test particle will have a nonzero velocity at one moment, it will never come to rest in the co-moving frame. Thus this frame may be considered to be a type of inertial frame. The everyday inertial and centrifugal forces which we experience on earth are precisely due to motion relative to this cosmic frame of reference, that of the so-called "fixed stars."

To determine completely the coordinate system, we also have to specify the spatial origin of the coordinates. Because of the equivalence of all points in three-space, this origin can be taken where most convenient,

depending on the particular problem considered, without affecting the form in which the line element is written. It will be specified in all forthcoming applications.

Next let us write the metric (12.49) in the more convenient form used by astronomers. In place of the radial distance r and the three-dimensional line element $d\sigma$, we define a dimensionless marker $u = r/r_0$ and a three-dimensional line element

$$d\chi^2 = \frac{1}{r_0^2}\, d\sigma^2 = du^2 + u^2(d\theta^2 + \sin^2\theta\, d\varphi^2) \tag{12.55}$$

Then, defining a new function $R^2(t) = r_0^2 e^{g(x^0)}$ with $t = x^0/c$, we can write

$$ds^2 = c^2\, dt^2 - R^2(t)\left[\frac{du^2 + u^2(d\theta^2 + \sin^2\theta\, d\varphi^2)}{[1 + (k/4)u^2]^2}\right] \tag{12.56}$$

Note that a given material point is considered to be labeled by co-moving coordinates u, θ, and φ, which are fixed in time, but that a physical distance interval between points, $R(t)\,\dfrac{d\chi}{1 + (k/4)u^2}$, is naturally time-dependent. However, the ratio of such a physical distance to $R(t)$, which we shall call the "radius" of the universe, is clearly time-independent, as one should expect.

In order to gain a more intuitive understanding of the metric form (12.56), let us digress somewhat for a moment to consider a problem in pure differential geometry. Suppose we have a four-dimensional Euclidean space with coordinates x^0, x^1, x^2, x^3 and a line element

$$ds^2 = (dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2 \tag{12.57}$$

If these coordinates are constrained to a hypersphere by the relation

$$(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2 = R^2 \tag{12.58}$$

then clearly

$$x^0\, dx^0 + x^1\, dx^1 + x^2\, dx^2 + x^3\, dx^3 = 0 \tag{12.59}$$

This allows us to eliminate dx^0 from the line element (12.57) and obtain instead

$$ds^2 = \sum_{i=1}^{3} (dx^i)^2 + \frac{\left[d \sum_{i=1}^{3} (x^i)^2\right]^2}{4(x^0)^2} \tag{12.60}$$

In terms of the usual spherical coordinates ρ, θ, φ defined by

$$x^1 = \rho \cos \varphi \sin \theta \qquad x^2 = \rho \sin \varphi \sin \theta \qquad x^3 = \rho \cos \theta \tag{12.61}$$

this becomes, since $R^2 = \rho^2 + (x^0)^2$,

$$\begin{aligned} ds^2 &= d\rho^2 + \rho^2(d\theta^2 + \sin^2 \theta \, d\varphi^2) + \frac{\rho^2 \, d\rho^2}{R^2 - \rho^2} \\ &= \frac{d\rho^2}{1 - \rho^2/R^2} + \rho^2(d\theta^2 + \sin^2 \theta \, d\varphi^2) \end{aligned} \tag{12.62}$$

With the convenient coordinate substitution

$$\rho = \frac{uR}{1 + \frac{1}{4}u^2} \tag{12.63}$$

Eq. (12.62) becomes

$$ds^2 = R^2 \left\{ \frac{du^2 + u^2(d\theta^2 + \sin^2 \theta \, d\varphi^2)}{(1 + \frac{1}{4}u^2)^2} \right\} \tag{12.64}$$

From the manner in which it has been constructed we may refer to this as the line element of a three-dimensional hypersphere of radius R which is imbedded in a four-dimensional Euclidean space.

It is now clear how the Robertson-Walker metric may be interpreted. Since (12.64) is identical with the space part of (12.56) for $k = 1$ we see that (12.56) describes an isotropic and homogeneous three-dimensional hypersphere with a uniform scalar curvature R given by

$$\frac{1}{R^2} = \frac{1}{R^2(t)} \tag{12.65}$$

This explains the name "radius of the universe," which we used for $R(t)$ (for $k \neq 1$, see Exercise 12.1). One must be careful to note that this geometric picture is without any observable consequences. The imbedding four-dimensional Euclidean space is entirely fictitious and serves only to enhance our intuitive understanding of the Robertson-Walker metric.

In the case $k = 1$, the hypersphere, the Robertson-Walker metric has some elegant geometrical properties worthy of further comment. To discuss these we introduce as a new radial coordinate the hyperspherical angle y, defined by

$$\sin y = \frac{u}{1 + u^2/4} \qquad dy^2 = \frac{du^2}{(1 + u^2/4)^2} \tag{12.66}$$

Since the range of u is 0 to ∞, the range of y is 0 to π. The Robertson-Walker metric in terms of y is

$$ds^2 = c^2 \, dt^2 - R^2(t)[dy^2 + \sin^2 y(\sin^2 \theta \, d\varphi^2 + d\theta^2)] \tag{12.67}$$

Let us now calculate the spatial distance from the origin of the coordinate system to the farthest point along a ray, $d\varphi = d\theta = 0$. This will be simply

$$\int_0^\pi R \, dy = \pi R \tag{12.68}$$

This may be identified as half the circumference of the hypersphere of radius R, an elegant result. We may similarly calculate the volume of a part of three-space at a given time; the invariant volume element of three-space will be

$$\sqrt{|g|} \, dy \, d\theta \, d\varphi = R^3 \sin^2 y \, dy \sin \theta \, d\theta \, d\varphi$$

as discussed in Sec. 3.2. Thus a sphere extending from $y = 0$ to y_0 has a volume given by

$$V(y_0) = 4\pi R^3 \int_0^{y_0} \sin^2 y \, dy = 4\pi R^3 \frac{2y_0 - \sin 2y_0}{4} \tag{12.69}$$

For very small y_0 this is $4\pi(Ry_0)^3/3$, which allows us to identify Ry_0 as a convenient measure of small radial distances. The total volume of the three-space is given for $y_0 = \pi$ as

$$V = 2\pi^2 R^3 \tag{12.70}$$

Lastly let us calculate the area of a sphere of coordinate radius y_0. Analogous to the volume element above, the surface area element will be $R^2 \sin^2 y_0 \sin^2 \theta \, d\theta \, d\varphi$. Thus the total area will be

That is, analogous to the marker r in the Schwarzschild metric, the *coefficient* $R^2 \sin^2 y$ of the angular interval $d\theta^2 + \sin^2 \theta \, d\varphi^2$ plays a distinguished role as a radial marker in terms of physical area; its square times 4π is the area of a sphere. (See Exercise 12.5 for interesting features of A.)

The cases of $k = 0$ and $k = -1$ may be thought of in geometrical terms also: $k = 0$ represents a flat Euclidean space with a time-dependent scale factor $R(t)$, while $k = -1$ represents a hypersurface of constant negative curvature, roughly analogous to the shape of the top of a saddle in two dimensions (see Exercises 12.1 to 12.4). Unfortunately a two-surface of constant negative curvature cannot be imbedded in Euclidean three-space, as recognized by Bolyai and Lobachevski, thus making it difficult to visualize, and similarly for a hypersurface of constant negative curvature.

In our derivation of the Robertson-Walker metric we postulated the existence of a universal time-coordinate and a Gaussian coordinate system in the large. Gödel (1949) obtained a cosmological solution of Einstein's equations which does not satisfy this postulate, but which does correspond to a constant matter density. There are difficulties associated with the physical interpretation of this solution, however. For example, the line element contains a cross term of the form $dx^0 \, dx^2$ and corresponds to an intrinsically rotating universe. We therefore defer discussion of this solution until the next chapter.

12.5 The Red Shift and the Robertson-Walker Metric: Hubble's Law

We shall now show that the Robertson-Walker metric (12.56) gives rise to an apparent shift in frequency of light emitted by distant objects, in agreement with the observed Hubble law. We consider a radiating object, say a galaxy G_e which is considered as a particle in our model. We then suppose that its light is observed by an observer in galaxy G_0 placed at the origin of the co-moving coordinates. The galaxy G_e is characterized by its distance marker u or, equivalently, by another useful *time-independent* marker l defined by the relation

$$dl = \frac{d\chi}{1 + (k/4)u^2} = \frac{[du^2 + u^2(d\theta^2 + \sin^2 \theta \, d\varphi^2)]^{1/2}}{1 + (k/4)u^2} \qquad (12.72)$$

In terms of the universal time used in the Robertson-Walker metric, the light emitted by G_e at time t_e is observed on G_0 at a time t_0 with $t_0 > t_e$.

Since light travels along a null geodesic, we must have $ds^2 = 0$, or

$$\int_{t_e}^{t_0} \frac{c\,dt}{R(t)} = l \qquad (12.73)$$

which relates the distance marker l to the time difference $t_0 - t_e$.

Consider now the light emitted by G_e at time $t_e + \Delta t_e$. It will be received by G_0 at time $t_0 + \Delta t_0$, where Δt_0 will be determined by the relation

$$\int_{t_e+\Delta t_e}^{t_0+\Delta t_0} \frac{c\,dt}{R(t)} = l \qquad (12.74)$$

since l is a fixed marker distance in the co-moving coordinates. Consider Δt_e as the period of some periodic physical phenomenon taking place on G_e, the emission of radiation for instance, and Δt to be short compared with the travel time from G_e to G_0. The periodic phenomenon will appear, as seen from G_0, to have a period Δt_0, which, from (12.73) and (12.74), will be such that the increment of the l integral is zero; thus, by elementary calculus,

$$\frac{\Delta t_0}{R(t_0)} - \frac{\Delta t_e}{R(t_e)} = 0 \qquad (12.75)$$

Calling $R(t_0) = R_0$ and $R(t_e) = R_e$ and writing (12.75) in terms of frequencies, we obtain

$$\frac{\nu_e}{\nu_0} = \frac{R_0}{R_e} \qquad (12.76)$$

For radiation propagating with velocity c, we associate a wavelength λ with a frequency ν, which, in view of the defining relation $c = \lambda\nu$, gives

$$\frac{\lambda_0}{\lambda_e} = \frac{R_0}{R_e} \qquad (12.77)$$

From this we define a relative shift in wavelength with respect to the wavelength at emission:

$$z = \frac{\Delta\lambda}{\lambda} = \frac{\lambda_0 - \lambda_e}{\lambda_e} = \frac{R_0}{R_e} - 1 \qquad (12.78)$$

From the above it is clear that the radiation emitted at one point will appear with a shift in wavelength at another point. This phenomenon discussed in Sec. 12.1. That is, λ_0 at reception is larger than λ_e at emission. Therefore, from (12.77), the function $R(t)$ must at present be monotonically increasing with time since the time of reception t_0 is later than the time of emission t_e. We thus must have a universe in expansion.

Hubble's law (12.2) asserts a linear relation between the red shift z and distance of a galaxy. We shall now derive this from (12.73) and (12.78) in first approximation. We begin by introducing a power-series expansion in terms of the travel time of light. Observe that $[R(t)]/c$ has the dimension of time and can be interpreted as the time needed by a light ray to reach a distance of the magnitude of the radius of the universe. We shall consider physical phenomena which require considerably smaller time intervals and use power-series expansions in terms of the quantity $[c(t_0 - t_e)]/R_0$, which can be considered as small. Formula (12.73) then tells us that l will be smaller than 1.

Expanding $1/[R(t)]$, we obtain

$$\frac{1}{R(t)} = \frac{1}{R_0} - \frac{R_0'}{R_0 c}\frac{(t-t_0)c}{R_0} + \frac{1}{c^2}\left[\frac{(R_0')^2}{R_0} - \frac{R_0''}{2}\right]\left[\frac{(t-t_0)c}{R_0}\right]^2 + O\left[\frac{c(t-t_0)}{R_0}\right]^3 \tag{12.79}$$

where the prime denotes differentiation with respect to t. Next, expanding (12.73) and (12.78) in powers of $[c(t_0 - t_e)]/R_0 = h$, we obtain

$$l = h + \frac{1}{2}\frac{R_0'}{c}h^2 + O(h^3) \tag{12.80}$$

and

$$z = \frac{R_0}{R_e} - 1 = \frac{R_0'}{c}h + \frac{R_0}{c^2}\left[\frac{(R_0')^2}{R_0} - \frac{R_0''}{2}\right]h^2 + O(h^3) \tag{12.81}$$

Eliminating h between the two equations above, we get

$$\begin{aligned} cz &= R_0'l + \frac{1}{2c}(R_0'^2 - R_0''R_0)l^2 + O(l^3) \\ &= R_0'l + \frac{R_0'^2 l^2}{2c}(1 + q_0) \end{aligned} \tag{12.82}$$

where

$$q_0 = -\frac{R_0''R_0}{R_0'^2} \tag{12.82$'$}$$

is referred to as the deceleration parameter because of its linear relation to $-R_0''$. This formula, which relates the red shift to the marker distance, will be of fundamental importance in establishing a theoretical relation between red shift and astronomical distance, both observable quantities to the astronomer.

Let us consider for the time being only the lowest-order term in (12.82) and remember that the physical distance which an astronomer measures is, in first approximation, $L \cong R_0 l$. Formula (12.82) then becomes

$$cz \cong \frac{R_0'}{R_0}(R_0 l) \cong \frac{R_0'}{R_0}L \tag{12.83}$$

This is precisely Hubble's red shift law (11.2) if we identify H with R_0'/R_0.

Hubble's law can also be interpreted as a Doppler effect on the basis of the Robertson-Walker metric (12.56). The galaxies G_0 and G_e have fixed co-moving coordinates, but the physical distance between them, $L = R(t)l$, increases with time. Indeed, the velocity is clearly

$$v = R'(t)l = \frac{R'(t)}{R(t)}L \tag{12.84}$$

Thus Doppler's law gives

$$\frac{\Delta\lambda}{\lambda} = \frac{v}{c} = \frac{R_0'L}{R_0 c} \tag{12.85}$$

and we again obtain Hubble's law (12.2).

In conclusion we can say that *the Robertson-Walker metric explains the red shift* and allows us to identify Hubble's "constant" in first approximation as

$$H = \frac{R_0'}{R_0} \tag{12.86}$$

Further considerations on the red shift will be taken up in the next section.

12.6 The Apparent Magnitude–Red Shift Relation

In the preceding section we considered the red shift as a function of the distance of an emitting galaxy and obtained the Hubble law as a first approximation. Unfortunately the measurement of the distance of a galaxy is an indirect and uncertain process, as we have already indicated. In contrast, we now consider the two parameters of a galaxy most easily measured, its red shift z and its apparent luminosity e_0, which is the amount of energy received from the star per unit time and unit area. We can relate these directly measurable parameters using only our knowledge of the geometry of space embodied in the Robertson-Walker metric; this section will indicate how this is done.

Let us use the usual polar coordinate system (u,θ,φ) and place the origin on the emitting galaxy G_e. An implicit relation exists between the red shift z and the marker u_0 of the observer's galaxy, which can in principle be obtained from (12.72), (12.73), and (12.78): from (12.73) and (12.72) we can relate u_0 to $R(t)$, and (12.78) relates z to $R(t)$. We may write this implicit relation symbolically as

$$A(z,u_0) = 0 \tag{12.87}$$

The next step of the program is to replace u_0 by the physically observable quantity e_0. Our knowledge of the Robertson-Walker metric allows us to do this on purely geometrical grounds if the total rate of emission E_e of the galaxy is known. We shall show that one can indeed obtain a relation of the form

$$B\left(\frac{e_0}{E_e}, u_0\right) = 0 \tag{12.88}$$

By elimination of u_0 between relations A and B we thus arrive at the desired relation, in symbolic form,

$$C\left(\frac{e_0}{E_e}, u_0\right) = 0 \tag{12.89}$$

This links the observable parameters e_0 and z if E_e is known. As we have discussed, astronomers know how to estimate E_e for some specific types of galaxies, so that (12.89) may be applied to a group of galaxies of similar type and therefore with roughly the same E_e.

Let us now carry out the above indicated operations and obtain an explicit and useful form of the apparent magnitude–red shift relation $C(e_0/E_e, z) = 0$. We need, first, to review a few elementary notions and definitions of astronomy. One defines a logarithmic scale to measure luminosities by defining the *apparent magnitude* of a star as

$$m_0 = -2.5 \log_{10} e_0 + \text{const} \qquad \text{(log to base 10)} \tag{12.90}$$

The zero of this logarithmic scale is fixed arbitrarily by defining the apparent magnitude of the North Star to be $m_0 = 2.15$. For an arbitrary star, one then has

$$m_0 = -2.5 \log_{10} \frac{e_0}{e_{\text{N.S.}}} + 2.15 \tag{12.91}$$

where $e_{\text{N.S.}}$ is the energy flux from the North Star. Such a procedure for measuring the brightness of a star finds its origin in the measurements made with the naked eye by early astronomers, who used an additive scale. It is now known that the response of the eye varies as the logarithm of the exciting intensity; hence the definition (12.91) of the apparent magnitude and its normalization to make the numbers agree with old catalogues of magnitudes. The luminosity of stars and galaxies is now measured more directly by means of instruments such as the bolometer, which gives e_0 directly. In practice one measures the red shift and the apparent magnitude of whole galaxies, and not of individual stars, for it is only for distant galaxies that the red shift due to the general recession is large enough to be measured and distinguished from the Doppler shift due to random velocities which are always present.

In order to calculate $C(e_0/E_e, z)$ explicitly, let us first derive the relation $B(e_0/E_e, u_0) = 0$ from the general form of the Robertson-Walker metric. Consider the total energy received per unit time on the whole surface of a sphere reached by light in a time $(t_0 - t_e)$. This energy is smaller than E_e for two reasons, which are best put in evidence by using the photon picture of light: First, the photons received have a degraded energy due to reddening by a factor $\nu_0/\nu_e = R_e/R_0$, where $R_e = R(t_e)$ and $R_0 = R(t_0)$. Second, if we consider the photons emitted at regular intervals Δt_e, they will arrive separated by longer intervals Δt_0, and the two rates will be in the ratio $\Delta t_e/\Delta t_0 = R_e/R_0$. Thus the energy received per unit time on the whole sphere surrounding the galaxy at a coordinate distance u_0 will be

$$E_0 = E_e\left(\frac{R_e}{R_0}\right)^2 \tag{12.92}$$

From the form of the Robertson-Walker metric in polar coordinates,

$$ds^2 = (dx^0)^2 - \frac{R(t)^2}{[1 + (k/4)u^2]^2}(du^2 + u^2\,d\theta^2 + u^2 \sin^2\theta\,d\varphi^2) \tag{12.93}$$

it is evident that a solid angle defined by $d\theta$ and $d\varphi$ corresponds at a coordinate distance u_0 to a physical area of

$$dS = \left(\frac{R_0 u_0}{1 + (k/4)u_0^2}\right) d\theta \left(\frac{R_0 u_0 \sin\theta}{1 + (k/4)u_0^2}\right) d\varphi \qquad R_0 = R(t_0) \tag{12.94}$$

Thus the total physical area of a sphere of coordinate radius u_0 is

$$S = \frac{4\pi R_0^2 u_0^2}{[1 + (k/4)u_0^2]^2} \tag{12.95}$$

The total energy received per unit area and unit time interval on the earth considered as a point on this sphere is therefore given by

$$e_0 = \frac{E_0}{S} = E_e \left(\frac{R_e}{R_0}\right)^2 \frac{[1 + (k/4)u_0^2]^2}{4\pi R_0^2 u_0^2} \tag{12.96}$$

This is precisely the quantity measured by astronomers using a bolometer. The apparent magnitude as defined by (12.91) thus becomes

$$m_0 = 5 \log_{10} \left[\frac{R_0 u_0}{R_e\left(1 + \frac{k}{4}u_0^2\right)}\right] - 2.5 \log_{10} E_e - \text{const} \tag{12.97}$$

Since the light coming from all stars is received at the same time of observation t_0, we have absorbed some $\log_{10} R_0$ factors in the constant in (12.97).

The ratio R_0/R_e appearing in (12.97) can be immediately expressed as a function of z through (12.78):

$$1 + z = \frac{R_0}{R_e} \tag{12.98}$$

This equation also gives in principle $t_0 - t_e$ as a function of z. From the fact that $ds^2 = 0$ along the path of a light ray, we can also obtain the expression $u_0/[1 + (k/4)u_0^2]$ as a function of t_e from (12.72) and (12.73):

$$\int_{t_e}^{t_0} \frac{c\,dt}{R(t)} = \int_0^{u_0} \frac{du}{1 + (k/4)u^2} \tag{12.99}$$

and then as a function of z, using (12.98). Therefore, once a specific function $R(t)$ is given, m_0 can be expressed as a function of z alone by using (12.99). That is, we may write

$$m_0 = 5 \log_{10} F(z) - 2.5 \log_{10} E_e + \text{const} \tag{12.100}$$

To give a simple and explicit illustration of the preceding paragraph, consider the case of an expanding Euclidean three-space, $k = 0$, and assume we are dealing with a model in which Hubble's law is strictly valid in the form

$$cz = HL \tag{12.101}$$

(see Sec. 13.6 also). Here L is the astronomical distance marker defined at the beginning of this chapter by the law of energy decrease with the inverse square of the distance. In this case, from the definition of L and the fact that the two-space is Euclidean, (12.96) can be replaced by

$$e_0 = \frac{E_e}{4\pi L^2} = \frac{E_e}{4\pi}\frac{H^2}{(cz)^2} \tag{12.102}$$

Therefore, by the use of (12.102), we see that (12.91) becomes, in this special case,

$$m_0 = 5 \log_{10} (cz) - 2.5 \log_{10} E_e + \text{const} \tag{12.103}$$

Indeed, this formula is the exact translation of Hubble's law in terms of the variables m_0 and z.

As a second example of how an explicit apparent magnitude–red shift relation is obtained, we shall consider the completely general relation (12.97) and assume that the quantities z, l, and u_0 are small. It is then possible to obtain a series expansion of m_0 in terms of z for an arbitrary function $R(t)$ (Robertson, 1955; Hoyle and Sandage, 1956). The calculation of the first two terms of m_0 is straightforward and requires that we retain only second-order terms in z and l throughout. Let us begin by writing (12.97) as

$$m_0 = 5 \log_{10} \frac{R_0}{R_e} + 5 \log_{10} \frac{u_0}{1 + (k/4)u_0^2} + \text{const} \tag{12.104}$$

where the constant depends upon the type of galaxy considered. Using (12.78), the first term is readily rewritten in terms of z, and we obtain

$$m_0 = 5 \log_{10}(1+z) + 5 \log_{10} \frac{u_0}{1+(k/4)u_0^2} + \text{const} \tag{12.105}$$

To express the second term as a function of z, first note that (12.72) can be easily integrated to give

$$l = \int_0^{u_0} \frac{du}{1+(k/4)u^2} = \frac{2}{\sqrt{k}} \arctan\left(\frac{\sqrt{k}}{2} u_0\right) \tag{12.106}$$

and thus

$$u_0 = \frac{2}{\sqrt{k}} \tan\left(\frac{\sqrt{k}\, l}{2}\right) \tag{12.107}$$

An elementary calculation then yields the result that the argument of the second logarithm in (12.105) is

$$\frac{u_0}{1+(k/4)u_0^2} = \frac{1}{\sqrt{k}} \sin(\sqrt{k}\, l) = l + O(l^3) \tag{12.108}$$

It thus only remains for us to express l as a series in z. To do this we invert Eq. (12.82) to obtain

$$l = \frac{cz}{R_0'}\left[1 - \frac{1}{2}\left(\frac{1+q_0}{R_0'}\right)z + \cdots\right] \tag{12.109}$$

We can thus express m_0 explicitly in terms of z by substituting (12.109) and (12.108) into (12.105):

$$\begin{aligned} m_0 = {} & 5 \log_{10}(1+z) + 5 \log_{10} cz \\ & + 5 \log_{10}\left[1 - \frac{1}{2}\left(\frac{1+q_0}{R_0'}\right)z + \cdots\right] + \text{const} \end{aligned} \tag{12.110}$$

where we have absorbed a new term, $-5 \log_{10} R_0'$, into the constant. This expression is easily expanded in terms of z; one must only be careful to remember that the logarithms are to base 10. The final result to first order in z is

$$\begin{aligned} m_0 &= 5 \log_{10} cz + \frac{5}{2 \log_e 10}(1-q_0)z + \text{const} \\ &= 5 \log_{10} cz + (1.086)(1-q_0)z + \text{const} \end{aligned} \tag{12.111}$$

which is completely general and holds for any function $R(t)$ having a reasonable enough behavior so that the power series we have used are valid.

One should notice that, to lowest order in z, the result (12.111) is identical with the previous simple relation (12.103).

Because of the logarithmic relation of m_0 to e_0 a plot of m_0 versus z (12.111) clearly does not give an observational value of H but does test the functional form of the Hubble law and yields a value for the deceleration parameter. Using the brightest galaxies in clusters, correcting for a variation of luminosity as a function of frequency, and allowing for a change in luminosity as a function of time, astronomers now estimate that (Sandage, 1972a, 1972b, 1972c)

$$q_0 = 1.0 \pm 0.5 \tag{12.112}$$

This value remains in doubt due to uncertainties in the corrections mentioned above, but it appears likely that q_0 is positive; from the definition (12.82′) this would mean that the universe is decelerating, $R_0'' < 0$. We can hope that the value of q_0 will eventually be determined to better accuracy with the use of radio galaxies, and perhaps also with the use of quasars if a classification scheme can be obtained to correlate absolute luminosities.

We now see that astronomical measurements are able to supply us with values for the Hubble constant H, the deceleration parameter q_0, and the density of material in the universe. In the following chapter on cosmological models we shall see how a knowledge of these parameters can be used to choose among various models and moreover how the Einstein equations can be used to relate q_0 and ρ_0 to k, thereby determining in principle the shape of the universe.

Exercises

12.1 In place of (12.63) we may introduce a coordinate substitution that formally encompasses the cases $k = 0$ and $k = -1$, as well as $k = 1$ as discussed in the text. Analogous to (12.63) we introduce

$$\rho = \frac{\sqrt{k}\, Ru}{1+(k/4)u^2} \qquad \sqrt{k}\, R = \tilde{R}$$

where $\tilde{R}$ is necessarily finite and real. Thus for $k = 1$ R must be real, for $k = -1$ it must be imaginary, and for $k = 0$ it must be infinite in the limiting sense that $\tilde{R}$ remains finite as $k \to 0$. Show that the three-space metric is

$$ds^2 = \tilde{R}^2 \left[\frac{du^2 + u^2(d\theta^2 + \sin^2\theta\, d\varphi^2)}{[1 + (k/4)u^2]^2}\right]$$

analogous to (12.64).

12.2 *(continued)* The $k = 1$ case can be "visualized" as a three-dimensional hypersphere imbedded in a fictitious four-dimensional space, as we showed in the text. How can the $k = 0$ case be visualized? Can the $k = -1$ case be visualized in an analogous way to $k = 1$? (This corresponds to the classical non-Euclidean geometry studied by Bolyai and Lobachevski. The three-hypersurface is referred to as a pseudo-hypersphere.)

12.3 Give a derivation of the Robertson-Walker metric from purely geometrical considerations based on plausibility arguments; use the three basic demands of Sec. 12.3. Do this by asking what two-dimensional surfaces are homogeneous and isotropic, then extend the question to three dimensions by analogy.

12.4 Analyze the $k = -1$, or pseudo-hyperspherical, universe in a similar manner to the hyperspherical case by introducing an angle defined by

$$\sinh y = \frac{u}{1 - u^2/4}$$

What are the analogues of the circumference of the universe (12.68), the volume expression (12.69), and the total volume of the hyperspherical universe (12.69′)?

12.5 From (12.71) plot A as a function of y_0. Note that it is maximum for $y_0 = \pi/2$, a point one-fourth of the way around the universe, and becomes zero for $y_0 = \pi$, the largest sphere that can be constructed. Give a physical or geometrical interpretation of this.

12.6 What is the Petrov type of a space-time with a Robertson-Walker metric for the three possible values of k? (See Exercise 10.8.)

Problems

12.1 In principle it is possible to make interferometric measurements of a star's position which are extremely accurate if the base line of the interferometer is sufficiently large. Show that the distance to which such methods could be used with a radio interferometer equal in size to the earth's orbital radius is of the order of 10^9 light years. (The earth's orbit is about 500 light seconds in radius and the wavelength of radio waves used could be of order 1 cm.) Such a system might be devised in practice using, for example, a satellite in solar orbit as one end of the interferometer and the earth as the other (see Weinberg, 1971, p. 429).

12.2 *(continued)* Discuss how the satellite interferometer system might work. Would it seem feasible to devise an optical interferometer of the scale of the solar system?

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CHAPTER

13

Cosmological Models

So far our considerations have been of a purely geometrical and rather general nature; that is, we have dealt with the form of $g_{\mu\nu}$ alone. There remains considerable latitude in the functional form of $R(t)$ and the values of the constants Λ and k which appear in the Robertson-Walker metric (12.56). We begin this chapter by pursuing the consequences of various specifications of these quantities. The consideration of these "normal" models will be followed by two brief sections on more "heretical" models: the steady-state model, which does not obey Einstein's equations, and the Gödel model, which does not have a Robertson-Walker type of metric. We conclude by discussing the converse of the apparent magnitude–red shift problem.

13.1 Einstein's Equations and the Robertson-Walker Metric

In the following sections we shall consider a fluid continuum of a highly idealized nature which consists of galactic clusters. This fluid will be described by an average density ρ and an average internal pressure p, both of which can be functions of time but not of space. From this we are led to a very simple general form for $T^{\mu}{}_{\nu}$. The co-moving coordinates of a galactic cluster on the average satisfy $\dot{x}^0 = 1$, $\dot{x}^1 = \dot{x}^2 = \dot{x}^3 = 0$, so from the fluid energy-momentum tensor (10.41) and the fact that $g_{00} = 1$ in the Robertson-Walker metric (12.56), we obtain

$$T^{\mu}{}_{\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & -\frac{p}{c^2} & 0 & 0 \\ 0 & 0 & -\frac{p}{c^2} & 0 \\ 0 & 0 & 0 & -\frac{p}{c^2} \end{pmatrix} \tag{13.1}$$

Note that this expression is clearly consistent with the Robertson-Walker metric since $T^1{}_1 = T^2{}_2 = T^3{}_3$ and $T^\mu{}_\nu = 0$ for $\mu \neq \nu$. The relative sizes of ρ and p/c^2 can be estimated by assuming, as is usual, that the pressure is due to the residual random motion of galactic clusters, that is, to local deviations from the average state. It is a well-known result of the kinetic theory of gases that the relation between pressure and density is $p/\rho = \hat{v}^2/3$, where $\hat{v}$ is the root-mean-square random velocity of the gas molecules. Applying this to the galactic-cluster gas of the universe, we see that

$$\frac{p/c^2}{\rho} = \frac{1}{3}\frac{\hat{v}^2}{c^2} \tag{13.2}$$

The observed random motions of galaxies are in general much less than c, so that, in most reasonable models of the universe, we expect the pressure component at the present epoch in the evolution of the universe.

From a comparison of the Robertson-Walker metric in the two forms (12.29) and (12.56) we see that

$$e^{G(x^0,r)} = \frac{R(t)^2}{r_0^2(1 + kr^2/4r_0^2)^2} \tag{13.3}$$

$$e^{g(x^0)} = R(t)^2 \qquad e^{f(r)} = \frac{1}{r_0^2(1 + kr^2/4r_0^2)^2}$$

Using $T^\mu{}_\nu$ in (13.1) and $G^\mu{}_\nu$ as obtained from a short calculation with (13.3) and (12.41), we can write the Einstein equations (12.4) in the form

$$\frac{8\pi\kappa}{c^2}\rho = -\Lambda - G^0{}_0 = -\Lambda + \left[\frac{3k}{R(t)^2} + \frac{3R'(t)^2}{c^2R(t)^2}\right] \tag{13.4a}$$

$$\frac{8\pi\kappa}{c^2}\left(\frac{p}{c^2}\right) = \Lambda + G^i{}_i = \Lambda - \left[\frac{k}{R(t)^2} + \frac{R'(t)^2}{c^2R(t)^2} + \frac{2R''(t)}{c^2R(t)}\right] \tag{13.4b}$$

For later use we shall now also obtain an equivalent system. By taking linear combinations we get two new equations with different structure; one contains only a second-order derivative, and the other is independent of Λ:

$$\frac{4\pi\kappa}{c^2}\left(\rho + \frac{3p}{c^2}\right) = \Lambda - \frac{3R''(t)}{c^2R(t)} \tag{13.5a}$$

$$\frac{4\pi\kappa}{c^2}\left(\rho + \frac{p}{c^2}\right) = \frac{k}{R^2(t)} + \frac{R'^2(t) - R(t)R''(t)}{c^2R^2(t)} \tag{13.5b}$$

We can already draw an important conclusion from these equations if we combine (13.4*a*) with (13.5*b*). We may bring (13.5*b*) into the form

$$\frac{d}{dt}\left[\frac{1}{c^2}\frac{R'(t)}{R(t)}\right] = \frac{k}{R(t)^2} - \frac{4\pi\kappa}{c^2}\left(\rho + \frac{p}{c^2}\right) \tag{13.6}$$

On the other hand, by differentiating (13.4*a*) with respect to time, we obtain

$$\frac{8\pi\kappa}{c^2}\frac{d\rho}{dt} = -\frac{6k}{R^3}R' + \frac{6R'}{R}\frac{d}{dt}\left(\frac{1}{c^2}\frac{R'}{R}\right) \tag{13.7}$$

Inserting (13.6) into (13.7) and rearranging, we obtain the differential identity

$$\frac{d}{dt}(\rho R^3) + \frac{p}{c^2}\frac{dR^3}{dt} = 0 \tag{13.8}$$

Let us take an element of volume in the co-moving coordinates. Its geometric content will be proportional to R^3, from the Robertson-Walker metric. Thus, if $V(t)$ denotes the actual volume content of the element considered, $M = \rho V$ would measure the mass content and (13.8) could be written as

$$dM + \frac{1}{c^2}p\,dV = 0 \tag{13.9}$$

Since c^2M measures the energy content of the element and $p\,dV$ the work done against the pressure forces, we see that the energy balance under the cosmic evolution is preserved. We may say equivalently that the entropy change under the evolution is zero; see Exercise 13.1.

If we prescribe the equation of state of the fluid, $p = p(\rho)$, we can obtain from (13.8) the dependence of R upon ρ. Indeed, we may write Eq. (13.8) in the alternative form

$$\frac{dR}{R} + \frac{1}{3}\left[\frac{d\rho}{\rho + p(\rho)/c^2}\right] = 0 \tag{13.10}$$

From this, the relation between R and ρ is obtained by simple integration. In the case of a pressure-free model, we find

$$R^3\rho = \text{const} \tag{13.11}$$

and in the case of the ideal gas law, $p = \alpha\rho$, we obtain

$$R^{3(1+\alpha/c^2)}\rho = \text{const} \tag{13.12}$$

In particular, if the pressure is entirely due to radiation, one has the well-known law $p = (c^2/3)\rho$ and the result $R^4\rho = \text{const}$.

At this point we would do well to note that ρ is only a *parameter*, which we may interpret as a *coordinate density* and not necessarily a physical density. Thus we cannot in general say that the mass M of a body is given in terms of the physical volume V by $M = \rho V$, except in the classical limit or in Euclidean space. In general the gravitational binding energy will contribute to M. This will be demonstrated and discussed further in Sec. 14.2. We can nevertheless be quite sure at this point that the physical mass of a portion of the universe will be *proportional* to its *volume*, due to the assumption of homogeneity.

Once we have found R as a function of ρ, or conversely, know $\rho = \rho(R)$, we may apply (13.4a) to find the time dependence of the radius R. These considerations show the freedom which we have in constructing cosmological models. We may prescribe the equation of state $p(\rho)$, the cosmological constant Λ, and the signature $k = \pm 1$ or 0. From this, all physical and geometric quantities are determined. However, we are not sure whether every model so obtained will be physically meaningful and appear in terms of real-valued functions.

It will be useful to summarize here the dimensionality of the different terms which appear in these equations: $R(t)$, which can be interpreted in a closed universe as a "radius," has the dimension of a length, Λ has the dimension of the inverse square of a length, and ρ is a mass density. Consequently, κ/c^2 has the dimension of a length divided by a mass, which is in agreement with its classical use. Theoretical physicists often like to reduce the number of dimensional constants used in a problem by arbitrarily setting some of them equal to 1 as long as compatibility permits. For instance, setting $\kappa/c^2 = 1$, we make mass and length units identical. Setting the speed of light $c = 1$ makes the time unit equal to the two others. All other constants of nature must then be expressible in terms of a single fundamental dimension; for instance, Planck's action constant has the dimension of the square of the fundamental dimension. We shall, however, remain explicit in our presentation and keep all constants in evidence, at the expense of a slight loss of brevity in the equations.

13.2 Static Models of the Universe

Formulas (13.4) and (13.5) give ρ and p as functions of $R(t)$. If we choose $R(t)$ to be a constant R independent of time, that is, a static universe, ρ and p will also be independent of time. Such an assumption does not lead to a red shift since it implies that $R_0 = R_e$ in (12.72). This makes any such model unsuitable for the description of the real physical universe unless one can explain the red shift from sources other than the space geometry. Moreover the blackbody radiation discussed in Sec. 12.1 finds no ready explanation in such a cosmological model. Let us, nevertheless, consider several particular cases of such models which are of historical interest.

Before entering into the mathematical development, we should explain why it is desirable to construct a model of a finite universe at all. Since the middle of the nineteenth century many physicists and astronomers had found disturbing paradoxes when they assumed an infinite and homogeneous universe. The simplest and most striking example is the famous Olbers paradox, which runs as follows: If we assume a matter distribution with constant average density throughout the universe and an equal behavior of matter at all points of the homogeneous universe, each volume element dV should emit radiation $\mu\, dV$ in all directions. Here μ is probably a very small constant, but it must be finite. It is also clear, according to classical notions, that the radiation density should decrease by an r^{-2} law, if r is the distance from the emitting volume element. Take, now, an observer at any given point of this universe. In a spherical shell of radius r and width dr around him lies a volume $4\pi r^2\, dr$, which radiates the total energy $\mu 4\pi r^2\, dr$. Of this, he receives only the amount proportional to $\mu(4\pi r^2\, dr)/r^2 = 4\pi\mu\, dr$. However, integrating over all values of r, we find that the total amount of energy radiated to our observer would be infinite. Thus the assumption of an infinite and uniform universe predicts an infinite brightness of the sky due to the summing of all contributions of the uniform matter distribution in the world. Similar paradoxes come from considerations of gravitational phenomena which are also mass-proportional and obey the inverse-square law. Hence, from the moment that Riemann indicated the possibility of a non-Euclidean geometry in a finite spherical world, the implications for cosmology were very seriously considered. In particular, since general relativity theory studies the close relations between geometry and matter, it was natural that, from the beginning, the question of finiteness of the universe should arise. We begin with the oldest and crudest models.

In the static case, Eqs. (13.4a) and (13.5a) become

$$\frac{8\pi\kappa}{c^2}\rho = -\Lambda + \frac{3k}{R^2} \tag{13.13a}$$

$$\frac{4\pi\kappa}{c^2}\left(\rho + \frac{3p}{c^2}\right) = \Lambda \tag{13.13b}$$

We can distinguish two cases, according to the value given to Λ.

Consider first $\Lambda = 0$; then, from (13.13b), the universe must be empty if we assume ρ and p to be nonnegative. Equation (13.13a) then implies that $k = 0$. That is, this case corresponds to a flat empty universe in which special relativity applies everywhere. On the other hand, to preserve the attractive possibility of a closed spherical universe ($k = 1$), one can also satisfy Eq. (13.13b) by assuming that the relation $\rho + 3p/c^2 = 0$ holds. This implies a large negative pressure $p = -\rho c^2/3$. The origin of such a negative pressure clearly cannot be easily explained as due to random residual velocities. It would therefore be necessary to postulate a new physical phenomenon with no observational justification to retain this model. Thus the model seems to be merely a mathematical construction of little physical significance. It leads, however, to a relation between R and ρ; from (13.13a)

$$R^2 = \frac{3kc^2}{8\pi\kappa\rho} \tag{13.14}$$

To keep the correct signature of the four-dimensional metric, R^2 must be positive, and therefore k must be $+1$. Thus one obtains a spherical universe of radius

$$R = c\left(\frac{3}{8\pi\kappa\rho}\right)^{1/2} \tag{13.15}$$

This yields $R \cong 10^{10}$ to 10^{11} light years for $\rho = 10^{-29}$ to 10^{-31} g/cm^3. In spite of its rather unphysical origin, this result for R is not physically unreasonable.

We now turn to the case $\Lambda \neq 0$, which gives more freedom in the parameters and allows a somewhat less artificial model. Although such a model probably cannot give a good description of the universe because of the absence of an intrinsic red shift, it is interesting to look at the order of magnitude that it predicts for Λ. Equations (13.4a) and (13.4b) give, for any positive pressure,

$$\frac{k}{R^2} < \Lambda < \frac{3k}{R^2} \tag{13.16}$$

Thus Λ has the order of magnitude of the inverse square of the "radius" of the universe. Therefore it cannot play an important role in the

description of phenomena inside the solar system, as we indicated in Sec. 12.2. Equation (13.5b) gives in the static case

$$R^2 = \frac{kc^2}{4\pi\kappa(\rho + p/c^2)} \tag{13.17}$$

Therefore k must be equal to $+1$. Since, physically, ρ is much larger than p/c^2, R is roughly the same as for $\Lambda = 0$ [Eq. (13.14)].

Historically, it was to obtain such testable predictions linking physical density and the size of the universe that Einstein introduced the cosmological constant Λ. Only later came Hubble's discovery and the derivation of the general form of the Robertson-Walker metric. At the same time nonstatic solutions of Einstein's equations were found which did not necessitate the introduction of the cosmological constant Λ. This was a great relief to Einstein, who never liked to make use of this constant which had the "logical right" to enter the equations, but which Einstein considered arbitrary and aesthetically undesirable.

Before closing this section, we should point out another strong objection to the static universe as described by Eqs. (13.13). It was pointed out by Lemaître (1931) that the static solutions to these equations are unstable. That is, under certain perturbations of the density ρ, the system would enter an evolution away from the original static state. Since small perturbations are always present in the real universe, the existence of an unstable static universe appears very improbable.

13.3 Nonstatic Models of the Universe

In nonstatic models one considers a time-dependent $R(t)$, and therefore one can expect in this case to be able to obtain a description of the red shift. Logically, any functional form for $R(t)$ is possible if it leads to a red shift and allows ρ and p to be positive. Thus, in particular, $R(t)$ must be an increasing function, at least for the present time and observable past times. Indeed, in first approximation, observation gives the value of the logarithmic derivative of $R(t)$ at the present time through (12.83). In nonstatic models we still have the freedom of making Λ equal to zero or not.

In the case $\Lambda = 0$, Einstein's equations (13.4a) and (13.5a) become

$$\frac{8\pi\kappa}{c^2}\rho = \frac{3}{R^2c^2}R'^2 + \frac{3k}{R^2} \tag{13.18a}$$

$$\frac{4\pi\kappa}{c^2}\left(\rho + \frac{3p}{c^2}\right) = -\frac{3R''}{c^2R} \tag{13.18b}$$

The first equation yields a very interesting relation between the present density of the universe, Hubble's constant, and the parameter k. Rearrangement gives

$$\rho - \frac{3H^2}{8\pi\kappa} = \left(\frac{3c^2}{8\pi\kappa R^2}\right) k \tag{13.18c}$$

since $H = R_0'/R_0$ and $\rho = \rho_0$ in the present epoch. Thus a knowledge of accurate values for ρ_0 and H would determine the sign of k, and therefore the hyperspherical, Euclidean, or pseudo-hyperspherical character of the three-dimensional world. With the presently accepted value of $H^{-1} = (5.6 \pm 0.6) \times 10^{17}$ s, the left side is $\rho_0 - (2.1 \pm 0.5) \times 10^{-29}$ g/cm^3. As we discussed in Sec. 12.1, ρ_0 is at present not well known observationally, and we can only make the weak statement $\rho_0 \gtrsim 10^{-31}$ g/cm^3; thus as yet we cannot use (13.19) to determine k. Therefore we shall consider solutions for all three possibilities in this section.

Under the reasonable assumption that ρ and p are positive, the second equation shows that $R'' < 0$; that is, the expansion must be decelerated and $R(t)$ cannot have a minimum or an inflection point. Clearly the deceleration parameter q_0 must be positive, which is in agreement with present observations. To be able to obtain a definite expression for $R(t)$ as a function of time, we shall make the physically reasonable assumption that the pressure term can be neglected. This appears to be observationally justified in the present epoch. Then, by elimination of ρ between Eqs. (13.18a) and (13.18b), we obtain a single differential equation for $R(t)$:

$$2RR'' + R'^2 + kc^2 = 0 \tag{13.19}$$

This equation is equivalent to

$$(RR'^2)' + kc^2R' = 0 \tag{13.20}$$

which admits the first integral

$$R'^2 = \frac{D_0 - kR}{R} c^2 \tag{13.21}$$

Inserting (13.21) into (13.19), we obtain

$$R'' = -\frac{D_0}{2R^2} c^2 \tag{13.22}$$

Thus the constant D_0 has to be positive to give $R'' < 0$ as required by (13.18b). We can determine D_0 by again using (13.18b), which gives

$$D_0 = \left(\frac{4\pi}{3} R^3\rho\right)\frac{2\kappa}{c^2} \tag{13.23}$$

The term in parentheses is clearly proportional to the total mass of the universe from (12.70) for the case $k = 1$ and is equal to the mass of a sphere of size R for the Euclidean case, $k = 0$. Moreover it is conserved during the evolution of the pressureless universe, as already noted in (13.11). We therefore label it, for all values of k, by the symbol $\hat{M}$. Thus (13.23) may be written as

$$D_0 = \frac{2\hat{M}\kappa}{c^2} \tag{13.24}$$

which is formally the same as the expression giving the Schwarzschild radius (see Sec. 6.8).

We now consider the solutions of (13.21) in the three cases $k = +1$, 0, -1. Consider first $k = +1$. By setting $R = D_0 \sin^2 \tau(t)$, we can solve (13.21) parametrically in terms of τ to obtain

$$ct = \frac{D_0}{2}[2\tau - \sin 2\tau] \qquad R = \frac{D_0}{2}[1 - \cos 2\tau] \tag{13.25}$$

The radius R as a function of t describes a cycloid (see Fig. 13.1). It begins at zero at $t = \tau = 0$, corresponding to a "big bang," or explosive birth of the universe; of course that initial epoch cannot be quantitatively well described by our present considerations since we are neglecting pressure. The radius then increases, reaches a maximum value of $D_0 = 2\kappa\hat{M}/c^2$ at $t = D_0\pi/2c$, and then contracts again to $R = 0$. Note the curious feature that the universe is always inside a Schwarzschild radius, defined in the present case as D_0; this should not be construed to mean that the density is always high (see Exercise 13.4).

The theoretical quantities D_0 and R (at the present epoch) can readily be related to the observable quantities H and q_0 for this model. From (13.21) and (13.22) and the definition of q_0 (12.82′) we obtain

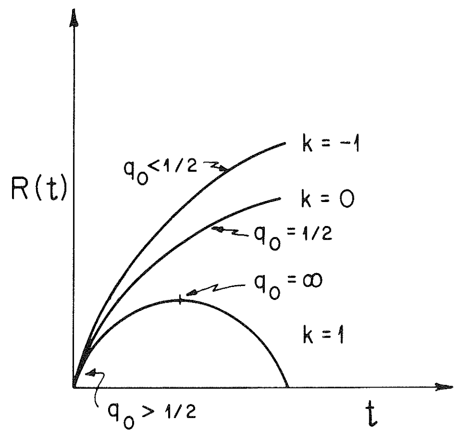

Fig. 13.1
Behavior of the dust-filled Robertson-Walker universe.

(13.26) $$2q_0 = \frac{D_0}{D_0 - R} \qquad \frac{R}{D_0} = 1 - \frac{1}{2q_0}$$

Thus q_0 ranges from $\frac{1}{2}$ for $R = 0$ to infinity at $R = D_0$, consistent with the observational value 1.0 ± 0.5. From the identification of H in (12.86) we then also obtain

(13.27) $$D_0 = \frac{2q_0}{(2q_0 - 1)^{3/2}} \frac{c}{H}$$

It is thus possible in principle to determine observationally all the properties of this model, which is indeed consistent with all present astronomical measurements (see Exercise 13.6).

Next let us consider the case $k = 0$. In this case (12.21) can be solved particularly easily, and we obtain

(13.28) $$R^{3/2} - R_0^{3/2} = \tfrac{3}{2}\sqrt{D_0}\, c(t - t_0)$$

This solution allows a radius equal to zero at a time

$$t_1 = t_0 - \frac{2R_0^{3/2}}{3c\sqrt{D_0}}$$

By shifting the time origin to t_1 we obtain the simpler form

(13.29) $$R = At^{2/3} \qquad A = (\tfrac{3}{2}c)^{2/3} D_0^{1/3}$$

This describes a universe in continuous expansion with an explosive birth at time $t = 0$, like the hyperspherical case (see Fig. 13.1). We can easily obtain the deceleration parameter from (13.21) and (13.22) as

(13.30) $$q_0 = \frac{-R_0''}{R_0 R_0'^2} = \frac{1}{2}$$

We cannot obtain an expression for D_0 in terms of the observables H and q_0 but may instead obtain an expression for ρ_0 in terms of H from (12.18*c*)

(13.31) $$\rho_0 = \frac{3H^2}{8\pi\kappa} = 2.1 \pm 0.5 \times 10^{-29} \text{ g/cm}^3$$

Since these numbers are not inconsistent with the presently determined values of q_0 and ρ_0, the model is viable at present. This solution, with $\Lambda = k = 0$, is the original nonstatic solution proposed by Friedman (1922). It was the first solution leading to a physically acceptable cosmological solution of Einstein's equations without a cosmological constant. Since it is desirable in theoretical physics to describe nature with a minimum number of universal constants, there is justification for considering this solution to be preferable to preceding ones.

We come now to the last case, $k = -1$. Equation (13.21) becomes, in this case,

(13.32) $$\frac{R'^2}{c^2} = \frac{D_0}{R} + 1$$

To solve this, we substitute, in analogy with the case $k = 1$,

$$R = D_0 \sinh^2 \tau(t)$$

Then we obtain a solution very similar to (12.25):

(13.33) $$ct = \frac{D_0}{2}(\sinh 2\tau - 2\tau) \qquad R = \frac{D_0}{2}(\cosh 2\tau - 1)$$

This shows that $R(t)$ increases monotonically from zero to infinity (see Fig. 13.1). Actually, this interesting fact could also have been read off from (13.32). Indeed, from (13.32) it is clear that R' remains of one sign and $R' > c$ if expansion starts. In fact, for large R, we see from (13.32) that the asymptotic behavior is

(12.34) $$R' = c \qquad R = ct$$

Therefore the model allows an expansion beyond D_0, the "Schwarzschild radius of the universe." The birth is explosive since $R' = \infty$ at $R = 0$, which is not quantitatively realistic due to the neglect of pressure. We can obtain interesting relations between D_0, R at the present epoch, q_0, and H, analogous to those obtained for the case $k = 1$. We leave it as an exercise to obtain the analogues of (13.26) and (13.27) and to show that $q_0 < \frac{1}{2}$. Such a value of q_0, and hence this model, is not inconsistent with the observed value of q_0.

In conclusion we may say that all the exploding models found in the case $\Lambda = 0$ are consistent with our present knowledge of the universe. An accurate observational value for q_0 could decide among the models since $q_0 > \frac{1}{2}$ for the hyperspherical model, $q_0 = \frac{1}{2}$ for the Euclidean model, and $q_0 < \frac{1}{2}$ for the pseudo-hyperspherical model; we can only say that $q_0 > \frac{1}{2}$ is somewhat favored by present data. The quantitative details of the explosive birth cannot be taken seriously due to the neglect of pressure. While it may be justifiable to neglect the pressure in the present state of the universe, this neglect is certainly not allowed for a highly condensed universe of small radius. However, if we kept the pressure term in the equations, we should have to specify the equation of state $p(\rho)$. It is difficult to say what this relation should be in a highly contracted universe, although there has been much speculation on the subject.

In the case $\Lambda \neq 0$ we have to deal with Eqs. (13.4) in all their generality. Originally, nonstatic solutions were introduced in order to allow one to dispense with the Λ-term in Einstein's equations. Therefore it is mainly for mathematical completeness that we mention here the possibility of nonstatic solutions, including the cosmological term. We shall present only one particular solution, which is of historical interest, having been the first nonstatic solution to be proposed.

This solution was given by de Sitter in 1917. It postulates that the rate of expansion of the universe is given exactly by the otherwise approximate relation

(13.35) $$\frac{R'}{R} = H$$

where H is Hubble's constant, as obtained in the limit of small astronomical distances. By elementary integration of (13.35), we obtain

(13.36) $$R = R_0 e^{Ht} \qquad R_0 = \text{const}$$

and the red shift versus parameter-distance relation (12.82) gives z proportional to l within $O(l^3)$. Such a model has no singularity. That is, $R(t)$ is neither zero nor infinite for finite times; it is unbounded in time and does not require a birth or a death. These properties give it a definite aesthetic appeal and mathematical convenience, but the consequences for ρ and p are drastic. Equations (13.5) reduce to

(13.37a) $$\frac{4\pi\kappa}{c^2}\left(\rho + \frac{3p}{c^2}\right) = \Lambda - \frac{3H^2}{c^2}$$

(13.37b) $$\frac{4\pi\kappa}{c^2}\left(\rho + \frac{p}{c^2}\right) = \frac{ke^{-2Ht}}{R_0^2}$$

The first equation indicates that $(\rho + 3p/c^2)$ is a constant, and the second indicates that $(\rho + p/c^2) \to 0$ for $t \to \infty$. Thus, with p and ρ positive, this model requires $\rho = p = 0$ at all times and also $k = 0$ from (13.37b). It therefore represents an empty universe with a time-dependent metric. It is Euclidean in space dimensions, but has an expanding scale. One can, however, still speak of a red shift if one thinks of a test atom and an observer situated in this otherwise empty universe; the wavelength of light emitted by the atom increases as the observer moves away from it. The actual universe might be considered as a set of local perturbations on a de Sitter geometry which is valid in the large. Locally, we could then solve the Schwarzschild problem and impose boundary conditions at infinity compatible with a de Sitter geometry.

The value of the cosmological constant is given by (13.37b):

(13.38) $$\Lambda = \frac{3H^2}{c^2} = \frac{1}{(1.8 \times 10^{10})^2}\ (\text{light years})^{-2}$$

It has the same order of magnitude as the value obtained in the static case, namely, the inverse square of the distance of the farthest objects seen today.

13.4 The Gödel Solution and Mach's Principle

In this section we disgress to discuss the Gödel solution, which does not have a Robertson-Walker type of metric. Our purpose is to present an interesting mathematical exercise and to illustrate in so doing that it is possible to find solutions of Einstein's equations without the simplifying mathematical assumptions used in Chap. 12; the exercise will also, we hope, clarify the relation between Mach's principle and general relativity theory.

Gödel (1949) showed that the following metric is compatible with an incoherent matter distribution:

$$ds^2 = (dx^0 + e^{\alpha x^1}\, dx^2)^2 - (dx^1)^2 - \tfrac{1}{2}e^{2\alpha x^1}(dx^2)^2 - (dx^3)^2 \tag{13.39}$$

The symbol α in this expression is a constant with the dimension of an inverse length. To verify that (13.39) is a solution we must compute the Christoffel symbols and the Einstein tensor corresponding to this metric. This is most easily done by obtaining the geodesic equations of (13.39). The Euler-Lagrange equations corresponding to the variational problem

$$\delta \int ds = 0 \tag{13.40}$$

are easily found to be

$$\begin{aligned} &\frac{d}{ds}(\dot{x}^0 + e^{\alpha x^1}\dot{x}^2) = 0 \\ -2&\frac{d}{ds}(\dot{x}^1) = 2\dot{x}^0\dot{x}^2\alpha e^{\alpha x^1} + (\dot{x}^2)^2\alpha e^{2\alpha x^1} \\ &\frac{d}{ds}\left[e^{\alpha x^1}\left(\dot{x}^0 + \frac{\dot{x}^2}{2}e^{\alpha x^1}\right)\right] = 0 \\ &\frac{d}{ds}(\dot{x}^3) = 0 \end{aligned} \tag{13.41}$$

By slight rearrangement we can bring (13.41) into the simpler form

$$\begin{aligned} &\ddot{x}^0 + 2\alpha\dot{x}^0\dot{x}^1 + \alpha e^{\alpha x^1}\dot{x}^1\dot{x}^2 = 0 \\ &\ddot{x}^1 + \alpha e^{\alpha x^1}\dot{x}^0\dot{x}^2 + \frac{\alpha}{2}e^{2\alpha x^1}(\dot{x}^2)^2 = 0 \\ &\ddot{x}^2 - 2\alpha e^{-\alpha x^1}\dot{x}^0\dot{x}^1 = 0 \\ &\ddot{x}^3 = 0 \end{aligned} \tag{13.42}$$

From this we can find the nonvanishing Christoffel symbols in the same manner as in Chap. 6. There results

$$\begin{Bmatrix} 0 \\ 0\ 1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1\ 0 \end{Bmatrix} = \alpha \qquad \begin{Bmatrix} 0 \\ 1\ 2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 2\ 1 \end{Bmatrix} = \frac{\alpha}{2}e^{\alpha x^1}$$

$$\begin{aligned} &\begin{Bmatrix} 1 \\ 0\ 2 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 2\ 0 \end{Bmatrix} = \frac{\alpha}{2}e^{\alpha x^1} \qquad \begin{Bmatrix} 1 \\ 2\ 2 \end{Bmatrix} = \frac{\alpha}{2}e^{2\alpha x^1} \\ &\begin{Bmatrix} 2 \\ 0\ 1 \end{Bmatrix} = \begin{Bmatrix} 2 \\ 1\ 0 \end{Bmatrix} = -\alpha e^{-\alpha x^1} \end{aligned} \tag{13.43}$$

and all others are zero.

We now use the definition of $R_{\mu\nu}$:

$$R_{\mu\nu} = \begin{Bmatrix} \kappa \\ \mu\ \kappa \end{Bmatrix}_{|\nu} - \begin{Bmatrix} \kappa \\ \mu\ \nu \end{Bmatrix}_{|\kappa} + \begin{Bmatrix} \kappa \\ \mu\ \lambda \end{Bmatrix}\begin{Bmatrix} \lambda \\ \kappa\ \nu \end{Bmatrix} - \begin{Bmatrix} \lambda \\ \kappa\ \lambda \end{Bmatrix}\begin{Bmatrix} \kappa \\ \mu\ \nu \end{Bmatrix} \tag{13.44}$$

Each of the four terms on the right side of (13.44) is a 4×4 matrix with only a few nonzero components. Observe first that $\begin{Bmatrix} \lambda \\ \kappa\ \lambda \end{Bmatrix} = \alpha$ for $\kappa = 1$ and zero for $\kappa \neq 1$. One then finds for the nonzero components the values

$$\begin{aligned} &\begin{Bmatrix} \kappa \\ 0\ 2 \end{Bmatrix}_{|\kappa} = \begin{Bmatrix} \kappa \\ 2\ 0 \end{Bmatrix}_{|\kappa} = \frac{\alpha^2}{2}e^{\alpha x^1} \qquad \begin{Bmatrix} \kappa \\ 2\ 2 \end{Bmatrix}_{|\kappa} = \alpha^2 e^{2\alpha x^1} \\ &\begin{Bmatrix} \kappa \\ 0\ \lambda \end{Bmatrix}\begin{Bmatrix} \lambda \\ 0\ \kappa \end{Bmatrix} = -\alpha^2 \qquad \begin{Bmatrix} \kappa \\ 2\ \lambda \end{Bmatrix}\begin{Bmatrix} \lambda \\ 2\ \kappa \end{Bmatrix} = \frac{\alpha^2}{2}e^{2\alpha x^1} \\ &\begin{Bmatrix} \lambda \\ \kappa\ \lambda \end{Bmatrix}\begin{Bmatrix} \kappa \\ 0\ 2 \end{Bmatrix} = \frac{\alpha^2}{2}e^{\alpha x^1} \qquad \begin{Bmatrix} \lambda \\ \kappa\ \lambda \end{Bmatrix}\begin{Bmatrix} \kappa \\ 2\ 2 \end{Bmatrix} = \frac{\alpha^2}{2}e^{2\alpha x^1} \end{aligned} \tag{13.45}$$

Substitution of these components into (13.44) immediately gives the contracted Riemann tensor

$$R_{\mu\nu} = -\alpha^2\begin{pmatrix} 1 & 0 & e^{\alpha x^1} & 0 \\ 0 & 0 & 0 & 0 \\ e^{\alpha x^1} & 0 & e^{2\alpha x^1} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{13.46}$$

The scalar $R^\mu{}_\mu$ is easily gotten by first inverting the metric tensor

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & e^{\alpha x^1} & 0 \\ 0 & -1 & 0 & 0 \\ e^{\alpha x^1} & 0 & \frac{1}{2}e^{2\alpha x^1} & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad \sqrt{-g} = \frac{e^{\alpha x^1}}{\sqrt{2}} \tag{13.47}$$

to get

$$(13.48) \qquad g^{\mu\nu} = \begin{pmatrix} -1 & 0 & 2e^{-\alpha x^1} & 0 \\ 0 & -1 & 0 & 0 \\ 2e^{-\alpha x^1} & 0 & -2e^{-2\alpha x^1} & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Raising one index of $R_{\mu\nu}$ and contracting then gives the simple result

$$(13.49) \qquad R^{\mu}{}_{\mu} = -\alpha^2$$

We have now calculated all the relevant geometric quantities connected with the metric (13.39).

Let us next compute the tensor $T_{\mu\nu}$. For an incoherent matter field at rest (we use co-moving coordinates), we know from Chap. 10 that

$$(13.50) \qquad T^{\mu\nu} = \rho v^{\mu} v^{\nu} = \rho \delta^{\mu}{}_{0} \delta^{\nu}{}_{0}$$

Lowering indices with the metric tensor (13.47), we then have

$$(13.51) \qquad T_{\mu\nu} = \rho g_{\mu 0} g_{\nu 0} = \rho \begin{pmatrix} 1 & 0 & e^{\alpha x^1} & 0 \\ 0 & 0 & 0 & 0 \\ e^{\alpha x^1} & 0 & e^{2\alpha x^1} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Note that this $T_{\mu\nu}$ is proportional to $R_{\mu\nu}$ in (13.46):

$$(13.52) \qquad R_{\mu\nu} = -\frac{\alpha^2}{\rho} T_{\mu\nu}$$

Let us now collect the results of our calculations. The Einstein equations with the cosmological term may be written in the form

$$(13.53) \qquad R_{\mu\nu} + (\Lambda - \tfrac{1}{2} R^{\beta}{}_{\beta}) g_{\mu\nu} = -\frac{8\pi\kappa}{c^2} T_{\mu\nu}$$

From (13.49) and (13.52) it is therefore evident that Gödel's metric is indeed a solution if

$$(13.54) \qquad \Lambda = -\frac{\alpha^2}{2} \qquad \frac{\alpha^2}{\rho} = \frac{8\pi\kappa}{c^2}$$

Note that the cosmological constant Λ is necessary to have a nontrivial Gödel solution.

In summary we see that the Gödel metric (13.39) satisfies Einstein's equations if the conditions (13.54) are satisfied; moreover, it is evident from (13.54) that if α is equal to zero, both Λ and ρ must be zero, which implies that space is flat (as we showed in Chap. 8). Thus α can be considered as a parameter which measures the deviation of the space from flatness. This fact will be useful later in this section.

Let us note at this point an interesting feature of our result. The matter of the universe is at rest in the coordinate system we have used, so the system is a universal co-moving system. This leads us to a surprising conclusion: The energy-momentum tensor (13.50) is indeed precisely the same as was used in connection with the Einstein static universe in Sec. 13.2. Thus the field equations

$$(13.55) \qquad R_{\mu\nu} + (\Lambda - \tfrac{1}{2} R) g_{\mu\nu} = C T_{\mu\nu} \qquad C = -\frac{8\pi\kappa}{c^2}$$

have two basically different solutions for the same $T_{\mu\nu}$. From the point of view of Mach's principle, we expect that the matter of the universe should uniquely determine the geometry of the universe. The situation noted above is clearly not consistent with this notion, which leads us to believe that Mach's principle is *not* built into general relativity via the field equations. Indeed, it appears that global notions (such as boundary conditions) must be added to general relativity to encompass Mach's principle. We shall say more about this later in the section.

We now wish to investigate some curious physical properties of the Gödel metric. First let us show that the world-lines which characterize matter at rest in Gödel's co-moving system (x^0,x^1,x^2,x^3) cannot be everywhere orthogonal to a one-parameter family of three-dimensional hypersurfaces. Note that this very important property of the metric intrinsically distinguishes the Gödel solution from any solution which admits a co-moving Gaussian coordinate system, that is, one for which $g_{00} = 1$ and $g_{0i} = 0$, such as the co-moving system of a Robertson-Walker metric. Indeed, if there existed such a family of hypersurfaces, one could mark off the s intervals of matter geodesics between hypersurfaces and thereby construct a co-moving Gaussian coordinate system by the method given in Sec. 2.4. It is thus evident that the above statement precludes the existence of a coordinate system in which there is a distinguished universal time-coordinate and in which the incoherent matter constituting the universe is at rest.

To demonstrate the statement of the preceding paragraph, suppose the contrary. That is, suppose that, imbedded in four-dimensional space, there is a family $\mathfrak{F}$ of three-dimensional hypersurfaces which are parametrized by λ and which have equations of the familiar form

(13.56) $$F(x^\mu) - \lambda = 0 \qquad (F = \text{fixed function})$$

If a vector dx^μ lies entirely within this surface, F will not change along it; that is, $dF = F_{|\mu}\, dx^\mu = 0$. Thus it is evident that one specific normal vector to the member of $\mathfrak{F}$ which contains the world-point x^μ is $F_{|\mu}(x^\mu)$. Thus any arbitrary vector field v_μ which is everywhere orthogonal to the members of $\mathfrak{F}$ may be written as

(13.57) $$v_\mu = lF_{|\mu}$$

where l is an arbitrary scalar function. With this in mind, let us construct from an arbitrary vector v^μ the completely antisymmetric tensor

(13.58) $$a_{\mu\nu\gamma} \equiv \{v_\mu v_{\nu|\gamma}\}$$
$$\equiv \frac{1}{3!}[v_\mu(v_{\nu|\gamma} - v_{\gamma|\nu}) + v_\nu(v_{\gamma|\mu} - v_{\mu|\gamma}) + v_\gamma(v_{\mu|\nu} - v_{\nu|\mu})]$$

For the special case of the vector field v_μ given by (13.57), this tensor is easily seen by direct calculation to be identically zero. Thus a covariant necessary condition that a vector field v_μ be everywhere orthogonal to a one-parameter family $\mathfrak{F}$ of three-dimensional hypersurfaces is

(13.59) $$a_{\mu\nu\gamma} = \{v_\mu v_{\nu|\gamma}\} = 0$$

For the case of the Gödel solution, we are interested in the specific vector which represents matter at rest:

(13.60) $$v^\mu = (1,0,0,0) \qquad v_\mu = (1,0,e^{\alpha x^1},0)$$

It is evident for this vector that

(13.61) $$v_{\nu|\gamma} = \begin{cases} \alpha e^{\alpha x^1} & \text{for } \nu = 2, \gamma = 1 \\ 0 & \text{otherwise} \end{cases}$$

and therefore that the tensor $a_{\mu\nu\gamma}$ is

(13.62) $$a_{\mu\nu\gamma} = \begin{cases} -\frac{1}{6}\alpha e^{\alpha x^1} & \text{even permutation of } 0, 1, 2 \\ \frac{1}{6}\alpha e^{\alpha x^1} & \text{odd permutation of } 0, 1, 2 \\ 0 & \text{otherwise} \end{cases}$$

Since $a_{\mu\nu\gamma}$ is not identically zero, we have completed our demonstration.

The tensor $a_{\mu\nu\gamma}$ which we have introduced above will also be useful in investigating the rotational behavior of a vector field. To see why the idea of rotation occurs in the first place in regard to the Gödel solution, let us recall a result of Chap. 4; we there obtained the metric corresponding to a flat space with a set of cylindrical coordinates r, φ, and z rotating about the z axis at a constant angular velocity w. The result (4.83) can be written as

(13.63) $$ds^2 = \left(1 - \frac{w^2r^2}{c^2}\right) c^2\, dt^2 - dr^2 - r^2\, d\varphi^2 - dz^2 - 2wr^2\, dt\, d\varphi$$

We now compare this with the Gödel metric (13.39) written as

(13.64) $$ds^2 = (dx^0)^2 - (dx^1)^2 - \tfrac{1}{2}e^{2\alpha x^1}(dx^2)^2 - (dx^3)^2 + 2e^{\alpha x^1}\, dx^0\, dx^2$$

There is an evident similarity between the forms of these two line elements. In particular, note that both have a cross term in the time interval. On the basis of this similarity, we shall make the tentative assumption that x^1 is a "radial" coordinate like r; x^2 is an "angular" coordinate like φ; and x^3 is an "axial" coordinate like z. (Actually Gödel carries out a transformation to a system r, φ, z, in which the correspondence between his line element and the "rotating flat space" line element (13.63) is made even more evident than above, but the computation is tedious and the result is not necessary for our purposes.) The formal similarity between (13.63) and (13.64) leads us to expect some sort of rotational behavior of the Gödel universe. Our problem is now to make such behavior explicit in a covariant way and justify the preceding intuitive notions.

To investigate the rotational character of the Gödel metric, let us use the tensor $a_{\mu\nu\gamma}$ introduced above to construct a new vector,

(13.65) $$\Omega^\beta = c\frac{\epsilon^{\beta\mu\nu\gamma}}{\sqrt{-g}}a_{\mu\nu\gamma} = c\frac{\epsilon^{\beta\mu\nu\gamma}}{\sqrt{-g}}\{v_\mu v_{\nu|\gamma}\}$$

We thereby associate a vector Ω^β with a given vector v_μ. Because of the presence of the antisymmetric tensor $\epsilon^{\beta\mu\nu\gamma}/\sqrt{-g}$ in the definition, one might expect Ω^β to be closely related to the ordinary curl of v_μ. This is indeed the case, as one can easily see by working out the components of Ω^β in a flat space with the usual coordinates of special relativity so that $\sqrt{-g} = 1$. The contravariant vector $\Omega^\beta = (\Omega^0, \mathbf{\Omega})$ associated with $v_\mu = (v_0, -\mathbf{v})$ is then found to be

$$\Omega^0 = -c\mathbf{v} \cdot (\nabla \times \mathbf{v})$$
(13.66)
$$\boldsymbol{\Omega} = -cv_0(\nabla \times \mathbf{v}) - c(\mathbf{v} \times \dot{\mathbf{v}}) - c(\mathbf{v} \times \nabla)v_0$$

To make the interpretation of Ω^β more transparent, consider the case of a vector field v_μ which represents a uniform field of counterclockwise rotation at angular velocity ω about the z axis of a three-dimensional Cartesian coordinate system x, y, z. That is, in these coordinates let

(13.67) $$v_0 = 1 \qquad \mathbf{v} = v^i = \left(\frac{\omega}{c}\,y, -\frac{\omega}{c}\,x, 0\right)$$

By immediate calculation, using (13.66), we find that

$$\Omega^0 = 0$$
(13.68)
$$\boldsymbol{\Omega} = -c(\nabla \times \mathbf{v}) = (0,0,2\omega)$$

Thus the uniform rotation is characterized by

(13.69) $$\Omega^\beta = (0,0,0,2\omega)$$

Now we apply the above interpretation of Ω^β to investigate the motion of co-moving matter in the Gödel universe. Note, first, that the covariant velocity vector v_μ of the co-moving matter field in a Robertson-Walker metric has the same components as the contravariant vector $v^\mu = (1,0,0,0)$, so both $a_{\mu\nu\gamma}$ and Ω^β are obviously zero for this field. However, for the Gödel metric, the matter field has an $a_{\mu\nu\gamma}$ given by (13.62), so one finds by direct calculation from (13.47) and (13.65) that

(13.70) $$\Omega^\beta = (0,0,0,\sqrt{2}\,\alpha c)$$

This vector has precisely the same form as (13.69), which characterizes a uniform rotation about the z axis. We are thus led to interpret the result (13.70) as indicating that the co-moving matter in the Gödel universe possesses a constant *intrinsic* angular velocity $\omega = \alpha c/\sqrt{2}$ about the x^3 axis. This statement can also be expressed in terms of more basic parameters of the Gödel universe by the use of (13.54) and (13.49):

(13.71) $$\omega = \frac{\alpha c}{\sqrt{2}} = 2\sqrt{\pi\kappa\rho} = c\left(\frac{-R^\mu{}_\mu}{2}\right)^{1/2}$$

The results of the preceding paragraph should leave the reader somewhat puzzled. We have shown formally that the co-moving matter of a Gödel universe undergoes an intrinsic uniform rotation. However, according to Mach's principle, the bulk matter of the universe should determine its geometry. Therefore, if the bulk matter of the universe is at rest in a particular coordinate system, one would expect that system to be inertial; for the Gödel universe this is not so. Furthermore, one must ask how the entire bulk matter of the universe can rotate; with respect to what does it rotate? To clarify these questions and put the problem in perspective, we shall study the motion of a test particle in a Gödel universe.

The equations of motion for a particle in a Gödel universe have already been obtained in (13.41). These equations can readily be partly integrated to give a new system

$$\dot{x}^0 + e^{\alpha x^1}\dot{x}^2 = a$$
(13.72)
$$e^{\alpha x^1}\left(\dot{x}^0 + \frac{\dot{x}^2}{2}\,e^{\alpha x^1}\right) = e^{\alpha x^1}\left(a - \frac{\dot{x}^2}{2}\,e^{\alpha x^1}\right) = b$$
$$\ddot{x}^1 + \alpha b\dot{x}^2 = 0 \qquad \dot{x}^1 + \alpha b x^2 = d$$
$$\dot{x}^3 = l$$

where a, b, d, and l are constants of integration. It is easy to see from these equations that a particle of matter at rest,

(13.73) $$x^\mu = (s,A,B,C) \qquad \dot{x}^\mu = (1,0,0,0) \qquad A, B, C \text{ const}$$

follows a geodesic for the following choice of constants:

(13.74) $$a = 1 \qquad l = 0 \qquad b = e^{\alpha A} \qquad d = \alpha b B$$

Let us ask what happens when a particle is not allowed to remain at rest but is given an initial velocity in the co-moving system. Specifically, suppose that we shoot a particle from the origin $x^i = 0$ in a radial direction toward some fixed target galaxy at the position $x^i = (A,0,0)$ (Fig. 13.2). The initial conditions are then

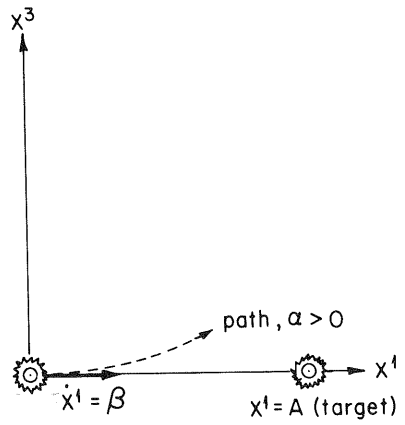

Fig. 13.2

$$x^0 = 0 \qquad \dot{x}^0 = 1 \qquad x^1 = 0 \qquad \dot{x}^1 = \beta$$
$$x^2 = 0 \qquad \dot{x}^2 = 0 \qquad x^3 = 0 \qquad \dot{x}^3 = 0 \tag{13.75}$$

which correspond to the constants

$$a = b = 1 \qquad d = \beta \qquad l = 0 \tag{13.76}$$

as one easily sees from Eqs. (13.72). Intuition and Mach's principle would then lead one to expect that, since matter determines geometry, the particle should travel in a straight line toward the distant target nebula. That is, the solution of (13.72) should have the form

$$x^\mu = (x^0(s), x^1(s), 0, 0)$$

We shall find, however, that this is not the case for the Gödel universe.

It is not difficult to solve the equations of motion (13.72) exactly for the initial conditions (13.75). However, the exact solution has a complicated form and is not very enlightening. We can learn all we need to know about the motion by supposing that α is a small parameter and then using perturbation theory. Recall that α is a measure of the nonflatness of space, so a perturbation approach is clearly well motivated. Let us begin by setting $\alpha = 0$ in (13.72). The solution of the resultant equations is elementary, and we obtain

$$x^\mu = (s, \beta s, 0, 0) \tag{13.77}$$

Thus, for flat space $\alpha = 0$, our conjecture in the preceding paragraph

is verified: The particle travels directly toward the distant target galaxy at constant coordinate velocity β.

Now suppose that α is small but nonzero, and expand equations (13.72) to first order in α:

$$\begin{aligned} \dot{x}^0 + \dot{x}^2 - 1 &= -\alpha x^1 \dot{x}^2 \\ \dot{x}^2 &= 2\alpha x^1(1 - \dot{x}^2) \\ \dot{x}^1 - \beta &= -\alpha x^2 \\ \dot{x}^3 &= 0 \end{aligned} \tag{13.78}$$

Into these equations we substitute the solution (13.77) plus a first-order correction αy^μ:

$$\begin{aligned} x^0 &= s + \alpha y^0 \\ x^1 &= \beta s + \alpha y^1 \\ x^2 &= \alpha y^2 \\ x^3 &= \alpha y^3 \end{aligned} \tag{13.79}$$

Using only first-order terms in α, this gives a set of equations for the y^μ:

$$\begin{aligned} \dot{y}^0 + \dot{y}^2 &= 0 \\ \dot{y}^2 &= 2\beta s \\ \dot{y}^1 &= 0 \\ \dot{y}^3 &= 0 \end{aligned} \tag{13.80}$$

Since the zeroth-order terms of (13.79) satisfy the initial conditions (13.75), all the y^μ and $\dot{y}^\mu$ are zero at $s = 0$. We thus arrive at the first-order solution

$$x^\mu = (s - \alpha\beta s^2, \beta s, \alpha\beta s^2, 0) \tag{13.81}$$

It is evident from (13.80) that the particle deviates from the ray

$$x^2 = x^3 = 0$$

and spirals outward instead of traveling directly toward the distant target galaxy, as it did for the case of flat space.

We are in a position to answer a few questions and come to some tentative and speculative conclusions concerning the Gödel universe.

First, we see that the matter of this universe does indeed rotate relative to "something"; the "something" is the path that a test particle follows if it is given an initial radial velocity. One may refer to the tangent of such a path as the "compass of inertia." Thus the compass of inertia rotates relative to the matter of the Gödel universe, or vice versa. Second, the interpretation of Mach's principle which we have mentioned above can be stated in more precise form: *The bulk matter* (or "fixed stars") *of the universe determines the compass of inertia, and the two cannot rotate relative to each other.* The Gödel solution, therefore, is consistent with general relativity theory, but not with this statement of Mach's principle; that is, general relativity does not imply Mach's principle as it is stated above. Thus it appears that ultimately Mach's principle must be incorporated into general relativity theory by the addition of some appropriate boundary or global condition.

13.5 The Steady-State Model of the Universe

In 1948 Bondi and Gold proposed a cosmological model which does not rely on Einstein's equations (Bondi and Gold, 1948). In their theory they do not postulate a specific link between matter and geometry in the form of a mathematical relation between $T^{\mu\nu}$ and $R^{\mu\nu}$, but instead postulate a constant energy density ρ and demand that an observer at any point in space-time find the same Hubble recession (12.2). All points in space-time thus have to be equivalent; this postulate is often called the "perfect cosmological postulate."

Put together, the above two requirements preclude the possibility of an energy conservation law. In fact, to keep the matter density ρ constant and to compensate for the expansion of the universe, it is necessary to assume that matter is created at a constant rate throughout space. To show this let us take an observer and draw a fixed sphere of radius r around him, using Euclidean geometry as an approximation. Because of the expansion of the scale of space, matter will flow out of this sphere at a rate $4\pi r^2 v\rho$, where v is the radial velocity of matter at the distance r. To compensate for this outflow, matter must be created inside the sphere at a constant rate $(4\pi/3)r^3 Q$, where Q is the average rate of creation of matter density. Thus, in a steady-state universe, one must have

$$4\pi r^2 \rho v = \frac{4\pi}{3} r^3 Q \tag{13.82}$$

Using Hubble's law, with $T = H^{-1}$,

$$v = \frac{r}{T} \tag{13.83}$$

we obtain

$$\dot{Q} = \frac{3\rho}{T} \tag{13.84}$$

which, translated into equivalent numbers of hydrogen atoms, gives $Q \simeq 10^{-(15\pm2)}$ hydrogen atoms/cm^3 year. Considering that ρ, so far as we know it, is within two orders of magnitude of 6×10^{-6} hydrogen atoms/cm^3, such a rate of matter creation certainly would be undetectable.

The main appeal of this model is that it has no singularity in its geometry; there is no beginning and no end of the universe. The constancy of T makes all times equivalent. However, the conceptual difficulty of a birth at the origin of time encountered in several earlier models is now replaced by the assumption of a continuous creation of matter, which may be equally hard to accept. To take this hypothesis seriously, we must also ask in what form this matter can be created. At this point it should be recalled that recent studies of Burbidge et al. (1957) have shown that the synthesis of elements in stars through successive nuclear reactions starting with free protons is theoretically possible; this theory explains the relative abundances of the elements in the universe rather well. In order that the steady-state theory remain consistent with these results, we therefore have to suppose that the matter is created in the form of hydrogen atoms, a somewhat awkward and artificial assumption.

Another serious objection to the steady-state theory is that it provides no ready explanation of the blackbody radiation discussed in Sec. 12.1. As a result the theory is now considered by most cosmologists as less viable than the evolving models discussed in Sec. 13.3. We include it here to illustrate how readily the mathematical structure of relativity theory can be altered to describe novel physical effects.

We shall now attempt to put the steady-state theory into a mathematical framework by the use of equations which are analogous to Einstein's equations without the cosmological term. It is impossible a priori to reconcile the steady-state theory with Einstein's equations as they now stand because of the energy conservation implied in these equations,

$$-\frac{8\pi\kappa}{c^2} T^{\mu\nu} = G^{\mu\nu} \tag{13.85}$$

since the right-hand side has a zero divergence. In the steady-state theory, the energy-momentum tensor cannot have a zero divergence because of the continuous creation of matter. However, one can write down equations of the same form as Einstein's equations by subtracting from $T^{\mu\nu}$ a tensor corresponding to the matter creation. This approach was proposed by Hoyle (1948), who wrote down modified Einstein equations of the form

$$-\frac{8\pi\kappa}{c^2}(T^{\mu\nu} - C^{\mu\nu}) = G^{\mu\nu} \tag{13.86}$$

Since $T^{\mu\nu}$ and $G^{\mu\nu}$ are symmetric, the tensor $C^{\mu\nu}$, which is thus introduced, must also be symmetric.

Equations (13.86) are not sufficient to determine completely the mathematical problem, so a subsidiary condition on the tensor $C^{\mu\nu}$ must be added. This can be done in the following more or less natural way. In the distinguished Gaussian coordinate system which we shall use, there is a naturally distinguished vector, $D^\mu = (1,0,0,0)$. Since the tensor $C^{\mu\nu}$ which is introduced in (13.86) is also a distinguished tensor in this Gaussian system, one might desire to relate D^μ and $C^{\mu\nu}$ to each other in order to keep the number of distinguished tensors to a minimum and thereby achieve a mathematical economy in the theory. The simplest way to relate a vector D^μ and a symmetric tensor $C^{\mu\nu}$ is via

$$C^{\mu\nu} = \tfrac{1}{2}A(D^{\mu\|\nu} + D^{\nu\|\mu}) \equiv \tfrac{1}{2}A(g^{\mu\alpha}D^{\nu}{}_{\|\alpha} + g^{\nu\alpha}D^{\mu}{}_{\|\alpha}) \tag{13.87}$$

where A is a constant scalar which we can later choose for our convenience. We thus add to the modified Einstein equations (13.86) the subsidiary requirement (13.87), with $D^\mu = (1,0,0,0)$. We shall see in the following paragraphs that the distinguished tensor thus defined does indeed put the steady-state theory into consistent mathematical form.

The postulates of homogeneity and equivalence of all world-points completely determine the metric. Indeed, the metric clearly must be of the Robertson-Walker type, and furthermore, because of the assumed constant rate of expansion of the universe, it must also be of the de Sitter type, with

$$R(t) = R_0 e^{t/T} \tag{13.88}$$

We shall, moreover, assume a priori that $k = 0$; that is, the three-dimensional space is Euclidean. Recall that the de Sitter universe can exist only for a vanishing energy density. In the present case, however, it is possible to obtain a model of the universe with a de Sitter type of metric and a nonvanishing constant matter density ρ, because of the presence of the matter-creation term in Eqs. (13.86). We shall write the Robertson-Walker metric in the form

$$ds^2 = dx_0^2 - \frac{R^2(t)}{R_0^2}(dx_1^2 + dx_2^2 + dx_3^2) \qquad x_0 = ct \qquad R(t) = R_0 e^{t/T} \tag{13.89}$$

Thus the metric tensor $g_{\mu\nu}$ and its inverse $g^{\mu\nu}$ are given by

$$g_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -\left(\frac{R}{R_0}\right)^2 & & \\ & & -\left(\frac{R}{R_0}\right)^2 & \\ & & & -\left(\frac{R}{R_0}\right)^2 \end{pmatrix}$$

$$g^{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -\left(\frac{R_0}{R}\right)^2 & & \\ & & -\left(\frac{R_0}{R}\right)^2 & \\ & & & -\left(\frac{R_0}{R}\right)^2 \end{pmatrix} \tag{13.90}$$

Let us now attack the mathematical problem of finding the most general tensor $C^{\mu\nu}$ such that the modified Einstein equations (13.86) hold for a universe with a Robertson-Walker metric of type (13.89) and in which there is a constant and uniform density ρ of matter which is anchored to co-moving coordinates. We shall later use the subsidiary condition (13.87) to single out the simplest and most appropriate choice of $C^{\mu\nu}$. We have, actually, a very easy task, since the tensor $G^{\mu\nu}$ is completely determined by the metric (13.89), and the energy-momentum tensor for incoherent matter anchored to co-moving coordinates is also uniquely given by $T^{\mu\nu} = \rho\delta^\mu{}_0\delta^\nu{}_0$. Thus the modified Einstein equations readily determine $C^{\mu\nu}$.

Let us compute $G^\mu{}_\nu$ first. Using $G^0{}_0$ and $G^1{}_1 = G^2{}_2 = G^3{}_3$ as conveniently displayed in (13.4) with the specific choices of $k = 0$, $R = R_0e^{t/T}$, we arrive at

$$G^\mu{}_\nu = -\frac{3}{c^2T^2}\delta^\mu{}_\nu \tag{13.91}$$

Then, by using $T^\mu{}_\nu = \rho\delta^\mu{}_0\delta^0{}_\nu$, we can solve the modified Einstein equations for $C^\mu{}_\nu$:

$$(13.92)\quad C^\mu{}_\nu = T^\mu{}_\nu + \frac{c^2}{8\pi\kappa}G^\mu{}_\nu = \begin{pmatrix} \rho - \frac{3}{8\pi\kappa T^2} & & & \\ & -\frac{3}{8\pi\kappa T^2} & & \\ & & -\frac{3}{8\pi\kappa T^2} & \\ & & & -\frac{3}{8\pi\kappa T^2}\end{pmatrix}$$

Raising one index with $g^{\mu\nu}$ given in (13.90), we also have $C^{\mu\nu}$:

$$(13.93)\quad C^{\mu\nu} = \begin{pmatrix} \rho - \frac{3}{8\pi\kappa T^2} & & & \\ & \left(\frac{R_0}{R}\right)^2 \frac{3}{8\pi\kappa T^2} & & \\ & & \left(\frac{R_0}{R}\right)^2 \frac{3}{8\pi\kappa T^2} & \\ & & & \left(\frac{R_0}{R}\right)^2 \frac{3}{8\pi\kappa T^2}\end{pmatrix}$$

We have not yet utilized the subsidiary condition (13.87). In order to apply it we first need to calculate the nonvanishing Christoffel symbols and then compute the tensor $D^\mu{}_{\|\nu}$. Using the Robertson-Walker metric (13.89), we find the nonzero Christoffel symbols as usual by obtaining the differential equations of the geodesic lines. The result is

$$(13.94)\quad \begin{aligned} \begin{Bmatrix} 0 \\ i\ j \end{Bmatrix} &= \frac{R'R}{cR_0^2}\delta_{ij} = \left(\frac{R}{R_0}\right)^2 \frac{1}{cT}\delta_{ij} \\ \begin{Bmatrix} i \\ 0\ j \end{Bmatrix} &= \frac{R'}{Rc}\delta_{ij} = \frac{1}{cT}\delta_{ij} \end{aligned}$$

The tensor $D^\mu{}_{\|\nu}$ is therefore

$$(13.95)\quad D^\mu{}_{\|\nu} = D^\mu{}_{|\nu} + \begin{Bmatrix} \mu \\ \alpha\ \nu \end{Bmatrix} D^\alpha$$

$$= \begin{pmatrix} 0 & & & \\ & \frac{1}{cT} & & \\ & & \frac{1}{cT} & \\ & & & \frac{1}{cT}\end{pmatrix}$$

Since both $D^\mu{}_{\|\nu}$ and $D^{\mu\|\nu}$ are clearly symmetric, we see from (13.87) that the subsidiary condition is $C^\mu{}_\nu = AD^\mu{}_{\|\nu}$; by comparison of (13.93) and (13.95), we thus obtain

$$(13.96)\quad \rho = \frac{3}{8\pi\kappa T^2} \qquad A = \frac{-3c}{8\pi\kappa T}$$

so that Hubble's constant is the only universal constant which still appears in the theory.

Lastly, let us investigate the divergence of $T^{\mu\nu} = \rho\delta^\mu{}_0\delta^\nu{}_0$, which in the present theory must be equal to the divergence of $C^{\mu\nu}$, since $G^{\mu\nu}$ is divergenceless. An easy calculation using the Christoffel symbols (13.94) gives

$$(13.97)\quad T^{\mu\nu}{}_{\|\nu} = T^{\mu\nu}{}_{|\nu} + \begin{Bmatrix} \mu \\ \alpha\ \nu \end{Bmatrix} T^{\nu\alpha} + \begin{Bmatrix} \nu \\ \alpha\ \nu \end{Bmatrix} T^{\alpha\mu} = \frac{3\rho}{cT}(1,0,0,0)$$

and

$$(13.98)\quad C^{\mu\nu}{}_{\|\nu} = C^{\mu\nu}{}_{|\nu} + \begin{Bmatrix} \mu \\ \alpha\ \nu \end{Bmatrix} C^{\nu\alpha} + \begin{Bmatrix} \nu \\ \alpha\ \nu \end{Bmatrix} C^{\mu\alpha} = \frac{3\rho}{cT}(1,0,0,0)$$

The two divergence vectors are therefore equal as the theory requires. There is another interesting feature of the result (13.98). One can interpret $T^{\mu\nu}{}_{\|\nu} = C^{\mu\nu}{}_{\|\nu}$ as the source vector of the energy-momentum tensor $T^{\mu\nu}$ in the same way as $s^\mu = F^{\mu\nu}{}_{\|\nu}$ is the source vector of the Minkowski tensor $F^{\mu\nu}$, as indicated in (4.58). The explicit form (13.98) for this vector therefore indicates that matter density is created at rest in the co-moving coordinate system with a rate $Q = 3\rho/T$. This is exactly the same relation as (13.84), which we found previously by heuristic arguments. Moreover, we can now relate the creation rate Q to Hubble's constant T by using the relation between ρ and T in (13.96):

$$(13.99)\quad Q = \frac{3\rho}{T} = \frac{9}{8\pi\kappa T^3}$$

In conclusion we can say that we have succeeded in setting up a consistent mathematical scheme for a steady-state cosmological theory with a Robertson-Walker type of metric and continuous creation of matter at a rate $Q = 3\rho/T$. The relation (13.96) between the density ρ and Hubble's constant is the same as that found by Friedman, according to (13.31), and is compatible with the present observational data as we indicated in Sec. 13.3.

13.6 Converse of the Apparent Magnitude–Red Shift Problem

In this section we shall consider the converse mathematical problem of Sec. 12.6: If one were given an accurate empirical relation between the apparent magnitude m_0 and the red shift z, what conclusions could be drawn concerning the geometry compatible with the relation? In particular, could one determine the function $R(t)$ of a corresponding Robertson-Walker metric directly from observational data?

As an illustrative example we select the case of the apparent magnitude–red shift relation (12.103), which we suppose now to be exact. This is equivalent to supposing Hubble's law to be exact, as we discussed in Sec. 12.6. From (12.97) and (12.103) we obtain by comparison

$$\frac{R_0}{R_e}\frac{u}{1+(k/4)u^2} = Bz \tag{13.100}$$

where B is a properly chosen constant. In view of (12.98) we can also assert that

$$\frac{u}{1+(k/4)u^2} = B\frac{z}{1+z} = \frac{B}{R_0}(R_0 - R_e) \tag{13.101}$$

On the other hand, we have a fundamental relation between R and u from (12.72), (12.73), and (12.106),

$$\int_{t_e}^{t_0}\frac{c\,dt}{R(t)} = \int_0^{u_0}\frac{du}{1+(k/4)u^2} = \frac{2}{\sqrt{k}}\arctan\left(\tfrac{1}{2}\sqrt{k}\,u_0\right) = \frac{2}{\sqrt{k}}\theta_0 \tag{13.102}$$

where the auxiliary variable $\theta = \arctan\left(\frac{1}{2}\sqrt{k}\,u\right)$ has been introduced for convenience. With this definition of θ we have

$$\sin 2\theta = \frac{2\tan\theta}{1+\tan^2\theta} = \frac{\sqrt{k}\,u}{1+(k/4)u^2} \tag{13.103}$$

Hence, (13.102) and (13.103) lead to the integral relation

$$\frac{u}{1+(k/4)u^2} = \frac{1}{\sqrt{k}}\sin\left\{\sqrt{k}\int_{t_e}^{t_0}\frac{c\,dt}{R(t)}\right\} \tag{13.104}$$

and (13.101) becomes a nonlinear integral equation:

$$R(t_0) - R(t_e) = \frac{R(t_0)}{B\sqrt{k}}\sin\left\{\sqrt{k}\int_{t_e}^{t_0}\frac{c\,dt}{R(t)}\right\} \tag{13.105}$$

This condition determines the value of $R(t)$ if we assume the precise relation (12.103) between apparent magnitude and red shift.

We have still the freedom to choose k and to prescribe the type of geometry desired. The simplest choice is evidently $k = 0$, which we shall pursue a little further. From the series development of $\sin\alpha$ it follows that (13.105) reduces in this case to

$$R(t_0) - R(t_e) = \frac{R(t_0)}{B}\int_{t_e}^{t_0}\frac{c\,dt}{R(t)} \tag{13.106}$$

which is a linear integral equation of much simpler form. It can even be reduced to an ordinary differential equation. We hold t_0 fixed and differentiate (13.106) on both sides with respect to the variable emission time t_e. We find, then, that

$$\frac{cR(t_0)}{B} = R(t_e)R'(t_e) = \frac{d}{dt_e}\left[\tfrac{1}{2}R(t_e)^2\right] \tag{13.107}$$

This has the obvious solution

$$R(t_e) = R(t_0)\left[1 - 2\frac{(t_0 - t_e)c}{BR(t_0)}\right]^{1/2} \tag{13.108}$$

where the arbitrary constant of integration has been chosen to fit the initial condition

$$R(t_e) = R(t_0) \qquad \text{for } t_e = t_0 \tag{13.109}$$

If we set $t_e = t_0$ in (13.107), we find that $c/B = R'(t_0)$, and using the approximate relation (12.86), we may connect the constant B with the Hubble constant H by the equation

$$BR(t_0) = \frac{c}{H} \tag{13.110}$$

Observe, however, that when $k = 0$, we have, by (13.101) and (12.72),

$$\frac{z}{1+z} = \frac{HR_0}{c} u = \frac{HR_0}{c} l \tag{13.111}$$

From this formula we can calculate the precise relation between the markers L as defined by Hubble's law $z = HL/c$ and l defined from the Robertson-Walker metric, instead of the first approximation $L = R_0 l$, which is frequently used in astronomy.

The Robertson-Walker geometry with $R(t)$ given by

$$R(t) = R(t_0)[1 + 2(t - t_0)H]^{1/2} \tag{13.112}$$

has several interesting features. We see that, at the moment $t_i < t_0$, determined by the condition

$$t_i = t_0 - \frac{1}{2H} \tag{13.113}$$

the radius of the universe is zero. If we measure time from this starting point, i.e., shift our time origin so that $t_i = 0$, we find $t_0 = 1/2H$ and the $R(t)$ law reduces to the simpler form

$$R(t) = R\left(\frac{1}{2H}\right)(2Ht)^{1/2} \tag{13.114}$$

This law implies an interesting pressure-density relation. Indeed, the Einstein field equations for $\Lambda = k = 0$ and for a Robertson-Walker metric imply, by (13.4) and (13.114),

$$\frac{8\pi\kappa}{c^2}\rho = \frac{3}{4c^2t^2} \qquad \frac{8\pi\kappa}{c^2}\frac{p}{c^2} = \frac{1}{4c^2t^2} \tag{13.115}$$

and thus the relation between pressure and density is

$$\rho = 3\left(\frac{p}{c^2}\right) = \frac{3}{32\pi\kappa t_e^2} \tag{13.116}$$

Recall, however, that the relation

$$\frac{p}{\rho} = \frac{c^2}{3} \tag{13.117}$$

characterizes electromagnetic radiation. Thus the "molecules" which comprise the "gas" of this universe would move at the speed of light and must therefore be photons, neutrinos, or other massless particles. It is generally supposed that p may be neglected compared to ρ in the present epoch, as we did in Sec. 13.3, but we really know very little about the density of such things as the massless particles in the universe, as mentioned in Sec. 12.1. We therefore cannot rule out the pressure-density relation (13.117). Moreover by comparing (12.103), on which this model is based, and (12.111) we see that $q_0 = 1$. This is consistent with the observational value of $q_0 = 1.0 \pm 0.5$. Thus this model is consistent with present data.

From the discussion in Sec. 13.3 and the above results we see that the present observational data are not yet adequate to allow us to reach definite conclusions about the validity of various models.

Exercises

13.1 In (13.9) we showed that the energy balance under cosmic evolution is preserved. Prove as stated in the text that entropy is also constant during the evolution.

13.2 Set up a system of natural units in which κ and c are both numerically equal to 1. One may then equate length and mass units. What mass corresponds to a length of 1 cm? What is the geometric mass $\kappa M/c^2$ of a proton, the earth, the sun, and the universe?

13.3 (*continued*) Discuss how one may work always in natural units without converting back to conventional units at the end of a theoretical derivation.

13.4 What is the approximate density of a system equal in size to its Schwarzschild radius if that radius is 1 cm (approximately the Schwarzschild radius of the earth), 3 km (approximately the Schwarzschild radius of the sun), and 10^{11} light years (approximately the Schwarzschild radius of the universe)?

13.5 A cycloid is the curve described by a point on the periphery of a rolling wheel; using this picture, give a geometrical interpretation of the parameter 2τ in (13.25) as an angle.

13.6 Use the value $q_0 = 1.0 \pm 0.5$ given in the text to obtain a numerical value for D_0 in the hyperspherical universe and a numerical value for

R in the present epoch. What are the observational limits on these quantities? How long will the universe last before $R = 0$ again? What is its present density?

13.7 Why can we not obtain D_0 and R in terms of H and q_0 for the Euclidean universe, $k = 0$?

13.8 Obtain analogues of (13.26) and (13.27) for the case $k = -1$, and also show that $q_0 < \frac{1}{2}$.

13.9 At what distance from us, in the nonstatic solutions, do receding galaxies reach the speed of light as observed at the present epoch? What is the observational effect, and is such a velocity of recession in consistent with any of the fundamental ideas of relativity?

Problems

13.1 What explanation of the red shift in Hubble's law could be given for a static nonevolving universe?

13.2 What explanation of the blackbody radiation could be given in a static nonevolving universe?

13.3 Show that the static universes defined by Eqs. (13.13) are unstable.

13.4 Discuss the physical density of a Robertson-Walker universe as opposed to the coordinate density ρ.

13.5 What is the Petrov type of the Gödel solution? (See Exercises 10.8 and 12.6).

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CHAPTER

14

The Role of Relativity in Stellar Structure and Gravitational Collapse

In Chap. 6 we discussed the metric of the exterior of a spherically symmetric distribution of mass, and in Chap. 7 we extended our considerations to include rotation. In this chapter we shall investigate the interior of massive bodies. For simplicity we limit ourselves to spherically symmetric systems with the special energy-momentum tensor (10.41), i.e., perfect fluids.

Our considerations have application in the study of stars which have reached the final stages of evolution. During most of the life of a star the light nuclei in the interior combine, i.e., undergo fusion, to release large quantities of energy. Much of this is in the form of radiation. This radiation produces a pressure that helps to counter the inward force of gravity, thereby stabilizing the star. For stars in which the fusion process has nearly ceased and little radiation pressure remains we may reasonably expect the stellar material to be approximately described by a perfect-fluid energy-momentum tensor in which phenomena such as viscosity and heat conduction are ignored. Such material, no longer capable of significant energy release via fusion, is generally referred to as *cold catalyzed matter:* it is cold in the sense that it behaves thermodynamically like a zero-temperature fluid and catalyzed in the sense that the fusion energy has nearly all been extracted.

After initial considerations on the basic equations of relativistic stellar structure for cold catalyzed matter we discuss the simple model of Schwarzschild, in which the proper density ρ is a constant. This will be followed by a discussion of the stability properties of very dense stars of cold catalyzed matter, which leads naturally to questions on the evolution of such stars. The simplest example of gravitational collapse, the spherical dust ball, will then be treated.